DIVISION BY ZERO CALCULUS (draft)

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Abstract: The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this book in a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we will introduce several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. We will see new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will collect many concrete properties in mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics.

Key Words: Division by zero, division by zero calculus, singularity, derivative, differential equation, $0/0 = 1/0 = z/0 = 0$, $\tan(\pi/2) = 0$, $\log 0 = 0$, $[(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} =$

1, infinity, discontinuous, point at infinity, Puha's horn torus model, Däumler's horn torus model, gradient, Laurent expansion, extension of solutions of differential equations, reduction problems of differential equations, analytic geometry, singular integral, conformal mapping, Euclidean geometry, Wasan, absolute function theory, Sato hyperfunction, distribution, generalized function, Isabelle/HOL, Riemann zeta function, axiom.

Okumura and Yoshinori:

Since the last version, please distribute it with the source file, openly.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

Preface

The division by zero has a long and mysterious history all over the world (see, for example, [14, 99] and the Google site with the division by zero) with its physical viewpoint since the document of zero in India in AD 628. In particular, note that Brahmagupta (598 -668 ?) established four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brāhmasphuṭasiddhānta. We have been, however, considering that his definition $0/0 = 0$ is wrong for over 1300 years, but, we will see that his definition is right and suitable.

The division by zero $1/0 = 0/0 = z/0$ itself will be quite clear and trivial with several natural extensions of fractions against the mysteriously long history, as we can see from the concept of the Moore-Penrose generalized inverse to the fundamental equation $az = b$, whose solution leads to the definition of $z = b/a$.

However, the result (definition) will show that for the elementary mapping

$$W = \frac{1}{z},$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotele. – For the related Greek philosophy, see [129, 130, 131]. However, as the division by zero we will consider the value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([57, 83]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, differential equations, complex analysis at the undergraduate level and to our basic idea for the space and universe.

We have to arrange globally our modern mathematics at our undergraduate level. Our common sense on the division by zero will be wrong, with our basic idea on the space and universe since Aristotele and Euclid. We would like to show clearly these facts in this book. The content is at an undergraduate level.

Close the mysterious and long history of division by zero that may be considered as a typical symbol of the stupidity of the human race and open the new world since Aristotele - Euclid.

December 2021 Kiryu, Japan Saburou Saitoh

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Division by zero calculus and differential equations.

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The author and Professor Hiroshi Okumura gave a lecture with the title of Wasan Geometry and Division by Zero at the RIMS research meetings on History of Mathematics at Kyoto University on 18-21, October 2018. The author was supported,

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The International Conference on Applied Physics and Mathematics, Tokyo, Japan, October 22-23.
http://www.meetingsint.com/conferences/
applied physics-mathematics
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The author was invited in The 6th Int'l Conference on Probability and Stochastic Analysis (ICPSA 2021), January 5-7, 2021 in Sanya, as a Keynote Speaker and a Technical Program Committee (TPC) member with the talk title:

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Professor Naoki Osada invited the author and Professor Hiroshi Okumura to his research meeting of the RIMS Joint Researches: Study of History of Mathematics on 2021.2.1 -

2021.2.3 at Kyoto University, RIMS. The author gavee the talk with the title:

History of division by zero

and Professor H. Okumura gave the talk with the title:

On the history of discourse of impossibility of division by zero from algebraic point of view

on 2020.2.1.14:00 - 2020.2.1.15:30.

They were supported by the RIMS.

We examined many and many undergraduate level textbooks in order to see the impacts of our division by zero, however, their contents are fundamental and therefore, we did not cite many books for the source materials.

Of course, I am very happy with my family Mariko Saitoh and Yoshinori Saitoh for the publication of this book. I would like to express my deep thanks to my family for their great contributions. In particular, Yoshinori examined and introduced a deal with information and related references on division by zero.

Meanwhile, we are having interesting negative comments from several colleagues on our division by zero. However, they seem to be just traditional and old feelings, and they are not reasonable at all for the author. The typical good comment for the first draft is given by a physicist as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

A typical wrong idea will be given as follows:

mathematical life is very good without division by zero (2018.2.8.21:43).

S. K. Sen and R. P. Agarwal [113] referred to our paper [51] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as a conclusion of the introduction of the book that:

"Thou shalt not divide by zero" remains valid eternally.

However, in [106] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

The compact version was published in [111] by the special favour of the publisher, promptly.

The detailed research procedures and many ideas on the division by zero are presented in the Announcements of the Institute of Reproducing Kernels as in cited in the references and Announcements in Japanese:

148(2014.2.12), 161(2014.5.30), 163(2014.6.17), 188(2014.12.15), 190(2014.12.24), 191(2014.12.27), 192(2014.12.27),

194(2015.1.2), 195(2015.1.3), 196(2015.1.4), 199(2015.1.15), 200(2015.1.16), 202(2.015.2.2), 215(2015.3.11), 222(2015.4.8), 225(2015.4.23), 232(2015.5.26), 249(2015.10.20), 251(2015.10.27), 253(2015.10.28), 255(2015.11.3), 257(2015.11.05), 259(2015.12.04), 262(2015.12.09),

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353(2017.2.2), 357(2017.2.17), 365(2017.5.12), 366(2017.5.16), 367(2017.5.18), 368(2017.5.19), 371(2017.6.27), 373(2017.7.17), 374(2017.7.20), 375(2017.7.21), 376(2017.7.31), 377(2017.8.3), 378 (2017.8.4), 398(2017.11.15), 399(2017.11.16), 402(2017.11.19). 404(2017.12.30), 405(2017.12.31),

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463(2019.1.2), 467(2019.1.3), 468 (2019.1.4), 470(2019.2.2), 477(2019.2.2), 479(2019.3.12), 480(2019.3.26), 481(2019.4.4), 483(2019.4.26), 484(2019.4.27), 485(2019.5.16), 490(2019.6.21), 493(2019.7.1), 495(2019.7.6), 496(2019.7.8), 497(2019.7.9), 498(2019.7.11), 499(2019.7.24), 500(2019.7.28), 505(2019.10.11), 506(2019.10.19), 508(2019.11.01), 510(2019.11.10), 512(2019.11.12), 516(2019.11.27), 517(2019.11.28), 518 (2019.11.29), 520(2019.12.04), 522(2019.12.08), 523(2019.12.09), 524(2019.12.10),

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Contents

1 INTRODUCTION

At first, we will recall a simple history of the division by zero.

1.1 Old global history of the division by zero

The global history of the division by zero is given by H. G. Romig ([99]) in details.

In short,

A. D. Brahmagupta (628): in general, no quotient, however, $0/0 = 0$.
Bhaskara (1152): $1/0 = \infty$.
John Wallis (1657) said that zero is no number and but$1/0 = \infty$, and he is the first to use the symbol ∞ for infinity.
John Craig (1716): impossible.
Isaac Newton (1744): the integral of dx/x is infinity.
Wolgang Boyai (1831): a/b has no meaning.
Martin Ohm (1832): should not be considered.
De. Morgan (1831): $1/0 = \infty$.
Rudolf Lipschtz (1877): not permissble.
Axel Harnack (1881): impossible.

Meanwhile, note that Euler stated that $1/0 = \infty$ ([35]). See the details:

Dividing by Nothing by Alberto Martinez:

> Title page of Leonhard Euler, Vollständige Anleitung zur Algebra, Vol. 1 (edition of 1771, first published in 1770), and p. 34 from Article 83, where Euler explains why a number divided by zero gives infinity.
>
> https://notevenpast.org/dividing-nothing/

N. Abel used $1/0$ as a notation of INFINTY:
https://ja.wikipedia.org/wiki/

For the paper [99], C. B. Boyer ([14]) stated that Aristotele (BC384 - BC322) considered firstly the division by zero in the sense of physics with many evidences and detailed discussions.

In fact, he stated strongly in the last part of the paper as follows:

> Tradition in this particular may prove to be trustworthy, but it necessarily must be rejected with respect to the more problem. Historical evidence points to Aristotele, rather than Btrahmaguputa, as the one who first considered division by zero.

However, in a strict sense, Brahmagupta (598 - 668 ?) introduced zero and he already defined as $0/0 = 0$ in Brhmasphuasiddhnta (628). However, our world history stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we showed that his definition is suitable. For the details, see the references.

India is great for mathematical sciences and philosophy, because basic arithmetic operations were discovered by Brahmagupta in 628 with zero, negative numbers and so on. However, his basic ideas were derived on the long history of India for void, nothing, infinity, non-existence and existence and so on. For example, in Vedas ([52]), we can find the decimal number system in very old days.

From the recent articles, we can study the related essential history. From [120, 121], we can see the long history of division by zero in India. For the great history of India for mathematics, we can see from [42, 113, 52].

In particular, we can see that Europian countries were very weak on ZERO and arithmetrics from, for example, [113].

Typically, F. Cajori ([15]) (1929) stated that Bernard Bdzano stated impossibility of the division by zero by showing a contradiction by the cancellation by zero. Meanwhile, C. W. Dodge ([30]) (1990) showed that from the algebraic viewpoint, the division by zero is impossible.

1.2 Recent situation of the division by zero

By a **natural extension** of fractions

$$\frac{b}{a} \tag{1.1}$$

for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{1.2}$$

incidentally in [104] by the Tikhonov regularization for the Hadamard product inversions for matrices, and we discussed their properties and gave several physical interpretations on the general fractions in [51] for the case of real numbers. The result is a very special case for general fractional functions in [18].

Indeed, we will show typical examples for $0/0 = 0$. However, in this introduction, these examples are based on some natural feelings and are not given as mathematics, because we do still not give the definition of $0/0$. However, following our new mathematics, these examples may be accepted as natural ones later.

The conditional probability $P(A|B)$ for the probability of A under the condition that B happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and from the meaning, $P(A|B) = 0$ and so, $0/0 = 0$.

For the representation of inner product $\mathbf{A} \cdot \mathbf{B}$ in vectors

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}},$$

if **A** or **B** is the zero vector, then we see that $0 = 0/0$. In general, the zero vector is orthogonal for any vector and then, $\cos\theta = 0$.

For this paragraph for our old version, Professor J. Czajko gave kindly the detailed comments following his some general idea for the division by zero under the multispatial reality paradigm and stated in the last part:

> As one can see, the single-space reality paradigm, which is unspoken in the former mathematics and physics, creates tacitly evaded inconsistencies even at the logical level of mathematical reasonings.
>
> Dieudonné ([32]) has also tentatively assumed $xy = 0$ wherever one of the variables is 0 and the other ∞ [*], which is similar to $0/0 = 0$. Besides, if your formula (1.2) would be rendered as $b/0 = 0 + i0$ then it might lead one to question whether or not the still reigning single-space reality paradigm is admissible in general.
>
> [*] Dieudonné J. Treatise on analysis II. NewYork: Academic Press, 1970, p.151.

Look his basic great references, [23, 24].

Meanwhile, W. Hövel said that
For the vector function

$$f(\mathbf{v}) = \frac{\mathbf{c} \cdot \mathbf{v}}{||\mathbf{v}||}$$

in the framework of integers, the result $f(\mathbf{0}) = 0$ is reasonable in his theory of bivectors, the Gauss circle problem; cellular automata and Moore neighborhoods (2020.10.4). For his theory see [43, 44].

For the differential equation

$$\frac{dy}{dx} = \frac{2y}{x},$$

we have the general solution with constant C

$$y = Cx^2.$$

At the origin $(0,0)$ we have

$$y'(0) = \frac{0}{0} = 0.$$

For three points a, b, c on a circle with its center at the origin on the complex z-plane with its radius R, we have

$$|a+b+c| = \frac{|ab+bc+ca|}{R}.$$

If $R=0$, then $a, b, c = 0$ and we have $0 = 0/0$.

For a circle with its radius R and for an inscribed triangle with its side lengths a, b, c, and further for the inscribed circle with its radius r for the triangle, the area S of the triangle is given by

$$S = \frac{r}{2}(a+b+c) = \frac{abc}{4R}.$$

If $R = 0$, then we have

$$S = 0 = \frac{0}{0}$$

(H. Michiwaki: 2017.7.28.). We have the identity

$$r = \frac{2S}{a+b+c}.$$

If $a+b+c=0$, then we have

$$0 = \frac{0}{0}.$$

Meanwhile, we obtain the differential equation:

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

When we fix $A = \pi/2$ and so $a = 2R$, we obtain

$$\frac{0}{0} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

Therefore, we have to have $\frac{0}{0} = 0$.

For the distance d of the centers of the inscribed circle and circumscribed circle, we have the Euler formula

$$r = \frac{1}{2}R - \frac{d^2}{2R}.$$

If $R = 0$, then we have $d = 0$ and

$$0 = 0 - \frac{0}{0}.$$

For the second curvature

$$K_2 = \left((x'')^2 + (y'')^2 + (z'')^2\right)^{-1} \cdot \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

if $(x'')^2 + (y'')^2 + (z'')^2 = 0$; that is, for the case of lines, then $0 = 0/0$.

For the function sign $x = x/|x|$, we have, automatically, sign $x = 0$ at $x = 0$.

We have furthermore many concrete examples as we will see in this book.

However, we do not know the reason and motivation of the definition of $0/0 = 0$ by Brahmagupta, furthermore, for the important case $1/0$ we can not find any result there. – Indeed, even nowadays, we can not find any good definition of the division by zero except our results and find many and many wrong logics on the division by zero, without the good definition of the division by zero $z/0$.

Professor H. Okumura ([72]) considered for Brahmagupta's idea as follows:

Indeed if $a \neq 0$, for the inversion a^{-1} of a

$$\frac{z}{a} = z \cdot a^{-1}.$$

Therefore in the case $a = 0$, for some X we assume that

$$\frac{z}{0} = z \cdot X.$$

However he could not specify the number X in this case. Hence he could not refer to $z/0$ for any number z. But the right side always equals 0 if $z = 0$ for any X. Thereby he could consider the following way:

$$\frac{0}{0} = 0 \cdot X = 0,$$

which implies that $0/0 = 0$. This seems to be the reason why he only referred to $0/0 = 0$.

S. Takahasi ([51]) discovered a simple and decisive interpretation (1.2) by analyzing the extensions of fractions and by showing the complete characterization for the property (1.2) in the following:

Proposition 1.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b,a)F(c,d) = F(bc,ad)$$

for all

$$a,b,c,d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a,b \in \mathbf{C}, a \neq 0.$$

Then, we obtain, for any $b \in \mathbf{C}$

$$F(b,0) = 0.$$

Note that the complete proof of this proposition is simply given by 2 or 3 lines, as we will give its complete proof later. In order to confirm the uniqueness result by Professor Takahasi, Professor Matteo Dalla Riva gave the proposition independently of Professor Takahasi as stated in ([51]). Indeed, when Takahasi's result was informed, he was first negative for the Takahasi uniqueness theorem.

In the long mysterious history of the division by zero, this proposition seems to be decisive. The paper had been published over fully six years ago, but we see still curious information on the division by zero and we see still many wrong opinions on the division by zero with confusions.

Indeed, Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following Proposition 1.1, we should **define**

$$F(b, 0) = \frac{b}{0} = 0,$$

and consider, for any complex number b, as (1.2); that is, for the mapping

$$W = f(z) = \frac{1}{z}, \tag{1.3}$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, their basic idea comes from

continuity with the common sense or based on the basic idea of Aristotele. – For the related Greece philosophy, see [129, 130, 131]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([57, 83]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic idea for the space and universe.

In addition, for the fundamental function (1.3), note that the function is odd

$$f(z) = -f(-z)$$

and if the function may be extended as an odd function at the origin $z = 0$, then the identity $f(0) = 1/0 = 0$ has to be satisfied. Further, if the equation

$$\frac{1}{z} = 0$$

has a solution, then the solution has to be $z = 0$.

Note that the identity

$$\int_0^\infty \sin(2\pi t\xi)d\xi = \frac{1}{2\pi}\frac{1}{t},$$

so, for $t = 0$, the both should be zero (H. Kobayashi: 2019.3.9.10:49). Of course, here the integral is considered in the sense of distribution theory.

Meanwhile, the division by zero (1.2) was derived from several independent ideas as in:

1) by the generalization of fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse to the fundamental equation $az = b$ that leads to the definition of the fraction $z = b/a$,

2) by the intuitive meaning (from the concept of repeated subtraction) of fractions (division) by H. Michiwaki,

3) by the unique extension of fractions by S. Takahasi, as in the above,

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,

and

5) by considering the values of functions with mean values of functions.

Furthermore, in ([56]) we gave the following results in order to show the reality of the division by zero in our world:

A) a simple field structure as the number system containing the division by zero — the **Yamada field Y**,

B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with its center at the origin on the complex z plane — the reflection point of zero is zero, (The classical result is wrong, see [83]),

and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

Furthermore, in ([57],[104]), we discussed many division by zero properties in the Euclidean plane - however, precisely, our new space is not the Euclidean space. In ([53]), we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero. More recently, we see the great impact to the Euclidean geometry in connection with Wasan in ([84, 68, 85, 86, 87, 88, 69]).

We will introduce a very beautiful horn torus model realizing our division by zero for the classical Riemann sphere from ([28]). We will be able to see pleasantly the whole world on the horn torus model that is the coincidence of zero point and the point at infinity by a conformal mapping from the extended complex plane onto the horn torus.

Our typical results were surprisingly confirmed by Isabelle/HOL system by José Manuel Rodriguez Caballero. We will refer the details in this book.

We will see the related basic references with division by zero.

J. A. Bergstra, Y. Hirshfeld and J. V. Tucker [11] and J. A. Bergstra [12] discussed the relationship between fields and the division by zero, and the importance of the division by zero for computer science. It seems that the relationship of the division by zero and field structures are abstract in their papers.

J. A. Bergstra (2019.7.29.19:15) gave his general survey:

> You can find the paper by searching for Transmathematica on google.
>
> If you search "division by zero a survey of options" in Google the paper appears at once,
>
> best wishes, Jan Bergstra

Meanwhile, J. Carlström ([16]) introduced the wheel theory;

> wheels are a type of algebra where division is always defined. In particular, division by zero is meaningful. The real numbers can be extended to a wheel, as can any commutative ring. The Riemann sphere can also be extended to a wheel by adjoining an element $\perp$, where $0/0 = \perp$. The Riemann sphere is an extension of the complex plane by an element ∞, where $z/0 = \infty$ for any complex $z \neq 0$. However, $0/0$ is still undefined on the Riemann sphere, but is defined in its extension to a wheel. The term wheel is introduced by the topological picture $\odot$ of the projective line together with an extra point $\perp = 0/0$.

Similarly, T. S. Reis and J.A.D.W. Anderson ([97, 98]) extends the system of the real numbers by defining division by zero with three infinities $+\infty, -\infty, \Phi$ (Transreal Calculus).

However, we can introduce simply a very natural field containing the division by zero that is a natural extension (modification) of our mathematics, as the Yamada field.

In connection with the deep problem with physics of the division by zero problem, see J. Czajko [23, 24, 25]. However, we will find many logical confusions in the papers, as we refer to the details later.

J. P. Barukčić and I. Barukčić ([9]) discussed the relation between the division 0/0 and the special relative theory of Einstein. However it seems that their result $0/0 = 1$ is curious with their logics. Their result contradicts with ours.

L. C. Paulson stated that I would guess that Isabelle has used this **convention** $1/0 = 0$ since the 1980s and introduced his book [64] referred to this fact. However, in his group the importance of this fact seems to be entirely ignored at this moment as we see from the book. He sent his e-mail as follows:

> There are situations when it is natural to define x/0 = 0. For example, if you define division using primitive recursion, in which all functions are total, you will get this identity. There is nothing deep about it.
>
> If you adopt this convention, it turns out that some identities involving division hold unconditionally, such as (x+y)/z = x/z + y/z. Other identities continue to require 0 to be treated separately, such as x/x = 1.
>
> The idea that x/0 = 0 is only a convention. It does not change mathematics in any significant way and it does not lead to contradictions either.
> (2017.07.04.00:22).

Jose Manuel Rodriguez Caballero introduced the information:

> you will find the discussion between Prof. Lawrence Paulson (https://www.cl.cam.ac.uk/ lp15/) and Prof. Harvey Friedman (https://math.osu.edu/people/friedman.8) concerning the division by zero.

For more recent idea on the division by zero, see L. C. Paulson ([91]). It seems that his group is not interested in the division by zero still.

See also P. Suppes ([117]) for the interesting viewpoint for the division by zero from the viewpoint of logic, pages 163-166.

For the recent great works, see E. Jeřábek and B. Santangelo [47, 112]. They stated in their abstracts of their papers as follows:

> E. Jeřábek [47]:
>
> For any sufficiently strong theory of arithmetic, the set of Diophantine equations provably unsolvable in the theory is algorithmically undecidable, as a consequence of the MRDP theorem. In contrast, we show decidability of Diophantine equations provably unsolvable in Robinson's arithmetic Q. The argument hinges on an analysis of a particular class of equations, hitherto unexplored in Diophantine literature. We also axiomatize the universal fragment of Q in the process.
>
> B. Santangelo [112]:
>
> The purpose of this paper is to emulate the process used in defining and learning about the algebraic structure known as a Field in order to create a new algebraic structure which contains numbers that can be used to define Division By Zero, just as i can be used to define $\sqrt{-1}$.

> This method of Division By Zero is different from other previous attempts in that each $\frac{\alpha}{0}$ has a different unique, numerical solution for every possible α, albeit these numerical solutions are not any numbers we have ever seen. To do this, the reader will be introduced to an algebraic structure called an S-Structure and will become familiar with the operations of addition, subtraction, multiplication and division in particular S-Structures. We will build from the ground up in a manner similar to building a Field from the ground up. We first start with general S-Structures and build upon that to S-Rings and eventually S-Fields, just as one begins learning about Fields by first understanding Groups, then moving up to Rings and ultimately to Fields. At each step along the way, we shall prove important properties of each S-Structure and of the operations in each of these S-Structures. By the end, the reader will become familiar with an S-Field, an S-Structure which is an extension of a Field in which we may uniquely define $\alpha/0$ for every non-zero α which belongs to the Field. In fact, each $\frac{\alpha}{0}$ has a different, unique solution for every possible α. Furthermore, this Division By Zero satisfies $\alpha/0 = q$ such that $0 \cdot q = \alpha$, making it a true Division Operation,

Meanwhile, we should refer to up-to-date information:

Riemann Hypothesis Addendum - Breakthrough Kurt Arbenz :

https://www.researchgate.net/publication/272022137 Riemann Hypothesis Addendum - Breakthrough.

Here, we recall Albert Einstein's words on mathematics:

> Blackholes are where God divided by zero. I don't believe in mathematics. George Gamow (1904-1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra"

> that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

We have still curious situations and opinions on the division by zero; in particular, two great challengers Jakub Czajko [24] and Ilija Barukčić [10] on the division by zero in connection with physics stated recently that we do not have the definition of the division 0/0, however $0/0 = 1$. They seem to think that a truth is based on physical objects and is not on our mathematics.

In particular, J. Czajko [26] stated in Section 9 as follows:

> Mathematics is mainly about forms and operations, and thus is truthless, but its objects must not only be consistent but also realistic, i.e. procedurally operational and structurally constructible. Yet presence of realistic operations and existence of constructible structures for the operations to be performed on the structures should be confirmed by experimental results. Mathematical truths cannot be established by abstract mathematical means alone. Yet the unconventional division by zero can reveal where the mathematical truth is about to vanquish due to unsubstantiated existential postulates or arbitrarily decreed operations. Mathematics must not be forced into submission by decrees, for enforcing nonsenses can backfire by producing faulty/contradictory conclusions, the acceptance of which can lead to failures.

In such a case, we will not be able to continue discussions on the division by zero more, because for mathematicians, they will not be able to follow their logic more. However, then we would like to ask for the question that what are the values and contributions of your articles and discussions. We will expect some contributions, of course.

This question will reflect to our mathematicians contrary. We stated for the estimation of mathematics in [102] as follows: Mathematics is a collection of relations and, good results are fundamental, beautiful, and give good impacts to human beings.

With this estimation, we stated that the Euler formula

$$e^{\pi i} = -1$$

is the best result in mathematics in details in:

No.81, May 2012 (pdf 432kb) www.jams.or.jp/kaiho/kaiho-81.pdf

At this point, we would like to recall that:

> Oliver Heaviside: Mathematics is an experimental science, and definitions do not come first, but later on.

In order to show the importance of our division by zero and division by zero calculus we are requested to show their importance with many examples. However, with the results stated in the references and in this book, we think that the importance of our division by zero is definitely stated clearly.

It seems that the long and mysterious confusions for the division by zero were based on its **definition**. – Indeed, when we consider the division by zero $a/0$ in the usual sense as the solution of the fundamental equation $0 \cdot z = a$, we have immediately the simple contradiction for $a \neq 0$, however, such cases $0/0$ and $1/0$ may happen, in particular, in many mathematical formulas and in many important physical formulas. The typical example is the case of $x = 0$ for the fundamental function $y = 1/x$.

– As we stated in the above, some researchers considered that for the mysterious objects $0/0$ and $1/0$, they considered them as ideal numbers as in the imaginary number i from its great success. However, such an idea will not be good as the number system, as we see simply from the concept of the Yamada field containing the division by zero.

Another important fact was **discontinuity** for the function $y = 1/x$ at the origin. Indeed, by the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the general fraction that is defined by the generalized solution of the equation $ax = b$. However, for the strong discontinuity of the function $y = 1/x$ at the origin, we were not able to accept the result $b/0 = 0$ for very long years.

As the number system containing the division by zero, the Yamada field structure is simple and perfect as a theory. However for the applications of the division by zero to **functions**, we will need the concept of **division by zero calculus** for the sake of uniquely determinations of results and for other reasons.

In this book, we will discuss the division by zero in calculus and Euclidean geometry and introduce various applications to differential equations and others, and we will be able to see that the division by zero is our elementary and fundamental mathematics.

In order to understand our long and wrong basic ideas for the point at infinity and the mirror image with respect to a circle, we refer to the properties of the stereographic projection and the mirror image in details in Sections 3 and 4.

In particular, we will introduce the Puha horn torus model and the Däumler's horn torus model that show our new world realizing our division by zero since Aristotele and Euclid.

This book is an extension of the source file ([96]) of the invited and plenary lecture presented at the International Conference – Differential and Difference Equations with Applications:

https://sites.google.com/site/sandrapinelas/icddea-2017

In this book, we would like to present clearly the conclusion of the talk:

The division by zero is uniquely and reasonably determined *as*

$$1/0 = 0/0 = z/0 = 0$$

in the natural extensions of fractions.

We have to change our basic ideas for our space and world.
We have to change globally our textbooks and scientific books on the division by zero.

2 INTRODUCTION AND DEFINITIONS OF GENERAL FRACTIONS

We first introduce several definitions of our general fractions containing the division by zero. This section will give the strong and natural background for our division by zero.

2.1 By the Tikhonov regularization

For any real numbers a and b containing 0, we will introduce general fractions

$$\frac{b}{a}. \tag{2.1}$$

We will think that for the fraction (2.1), it will be given by the solution of the equation

$$ax = b.$$

Here, in order to see its essence, we will consider all on the real number field $\mathbf{R}$. However, for $b \neq 0$, since $0 \times x = 0$ this equation does not have any solution for the case $a = 0$, and so, people thought that the division by zero is, in general, impossible for long years. In order to consider some general concept for the division, we will need some new idea.

At first, by the concept of the Tikhonov regularization method, we will consider the equation as follows:

For any fixed $\lambda > 0$, the minimum member of the Tikhonov function in x

$$\lambda x^2 + (ax - b)^2;$$

that is,

$$x_\lambda(a, b) = \frac{ab}{\lambda + a^2}$$

may be considered as the fraction in the sense of Tikhonov with parameter λ, in a generalized sense. Note that the limit

$$\lim_{\lambda \to +0} x_\lambda(a, b)$$

exists always. By the limit

$$\lim_{\lambda \to +0} x_\lambda(a,b) = \frac{b}{a}, \tag{2.2}$$

we will define the general fractions b/a.

Note that, for $a \neq 0$, the definition (2.2) is the same as in the ordinary sense. However, since $x_\lambda(0,b) = 0$, always, for $a = 0$, we obtain the desired results $b/0 = 0$.

The result (2.2) is, of course, a trivial Moore-Penrose generalized inverse (solution) for the equation $ax = b$. The Moore-Penrose generalized inverse gives very natural and generalized solutions for some general linear equations and its theory is well-established as a classical one. In this sense, we can say that our division by zero is trivial and clear against the long and mysterious history of the division by zero.

Indeed, we will be able to see that our division by zero is known by the Moore-Penrose generalized inverse. We will recall its essence.

For a complex number α and the associated matrix A, the correspondence

$$\alpha = a_1 + ia_2 \longleftrightarrow A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

is homomorphism between the complex number field and the matrix field of 2×2.

For any matrix A, there exists a uniquely determined Moore-Penroze generalized inverse $A^\dagger$ satisfying the conditions, for complex conjugate transpose $*$,

$$AA^\dagger A = A,$$

$$A^\dagger A A^\dagger = A^\dagger A,$$

$$(AA^\dagger)^* = AA^\dagger,$$

and

$$(A^\dagger A)^* = A^\dagger A,$$

and it is given by, for $A \neq O$, not zero matrix,

$$A^\dagger = \frac{1}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \cdot \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A = O$, then $A^\dagger = O$.

In general, for a vector $x \in \mathbf{C}^n$, its Moore-Penrose generalized inverse $x^\dagger$ is given by

$$x^\dagger = \begin{cases} 0^* & \text{for } x = 0 \\ (x^* x)^{-1} x^* & \text{for } x \neq 0. \end{cases}$$

For the general theory of the Tikhonov regularization and many applications, see the cited references, for example, [110].

2.2 By the Takahasi uniqueness theorem

S. Takahasi ([118]) established a simple and natural interpretation (2.1) by analyzing any extensions of fractions and by showing the complete characterization for the generalized fractions (2.1). Furthermore, he examined several fundamental properties of the general fractions from the viewpoint of operator theory. See [118]. His result will show that the result (2.1) should be accepted as a natural one.

Theorem 2.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a, b)F(c, d) = F(ac, bd)$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(a, b) = \frac{a}{b}, \quad a, b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a,0) = 0.$$

Proof. Indeed, we have

$$F(a,0) = F(a,0)1 = F(a,0)\frac{2}{2} = F(a,0)F(2,2) =$$

$$F(a \cdot 2, 0 \cdot 2) = F(2a,0) = F(2,1)F(a,0) = 2F(a,0).$$

Thus $F(a,0) = 2F(a,0)$ which implies the desired result $F(a,0) = 0$ for all $a \in \mathbf{C}$.

Several mathematicians pointed out to the author for the publication of the paper ([51]) that the notations of 100/0 and 0/0 are not good for the sake of the generalized sense, however, there does not exist other natural and good meaning for them. Why should we need and use any new notations? Any new notation will create complicated notations and confusions for fractions, as we see from this book. Indeed, we will see in this book that many and many fractions in our formulas will have this meaning with the concept of the division by zero calculus for the case of functions.

We have still curious confusions for the division by zero. Their basic reason will be given by that we were not able to give any reasonable definition of the division by zero.

2.3 By the Yamada field containing the division by zero

As an algebraic structure, we will give the simple field structure containing the division by zero.

We consider

$$\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$$

and the direct decomposition

$$\mathbf{C}^2 = (\mathbf{C} \setminus \{0\})^2 + (\{0\} \times (\mathbf{C} \setminus \{0\})) + ((\mathbf{C} \setminus \{0\}) \times \{0\}) + \{0\}^2 .$$

Then, we note that

Theorem 2.2 *For the set* $\mathbf{C}^2$*, we introduce the relation* $\sim$*: for any* $(a, b), (c, d) \in (\mathbf{C} \setminus \{0\})^2$*,*

$$(a, b) \sim (c, d) \Longleftrightarrow ad = bc$$

and, for any $(a, b), (c, d) \notin (\mathbf{C} \setminus \{0\})^2$*, in the above direct decomposition*

$$(a, b) \sim (c, d).$$

Then, the relation $\sim$ *satisfies the equivalent relation.*

Definition 2.1 *For the quotient set by the relation* $\sim$ *of the set* $\mathbf{C}^2$*, we write it by* A *and for the class containing* (a, b)*, we shall write it by* $\frac{a}{b}$*.*

Note that

Lemma 2.1 *In* $\mathbf{C}^2$*, if* $(a, b) \sim (m, n)$ *and* $(c, d) \sim (p, q)$*, then* $(ac, bd) \sim (mp, nq)$*.*

Then, we obtain the main result, as we can check easily:

Theorem 2.3 *For any members* $\frac{a}{b}, \frac{c}{d} \in A$*, we introduce the product* $\cdot$ *as follows:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and the sum $+$*:*

$$\frac{a}{b} + \frac{c}{d} = \begin{cases} \frac{c}{d}, & \text{if } \frac{a}{b} = \frac{0}{1}, \\ \frac{a}{b}, & \text{if } \frac{c}{d} = \frac{0}{1}, \\ \frac{ad+bc}{bd}, & \text{if } \frac{a}{b}, \frac{c}{d} \neq \frac{0}{1}, \end{cases}$$

then, the product and the sum are well-defined and A *becomes a field* $\mathbf{Y}$*.*

Proof. Indeed, we can see easily the followings: 1) Under the operation $+$, $\mathbf{Y}$ becomes an abelian group and $\frac{0}{1} = 0_Y$ is the unit element.

2) Under the operation $\cdot$, $\mathbf{Y} \setminus \{0_Y\}$ becomes an abelian group and $\frac{1}{1}$ is the unit element.

3) In $\mathbf{Y}$, operations $+$ and $\cdot$ satisfy distributive law.

Remark. *In $\mathbf{C}^2$, when $(a,b) \sim (m,n)$ and $(c,d) \sim (p,q)$, the relation $(ad+bc, bd) \sim (mq+np, nq)$ is, in general, not valid. In general,*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

is not well-defined and is not valid.

Indeed, $(1,2) \sim (1,2)$ and $(3,0) \sim (0,3)$, but

$$(1 \cdot 0 + 2 \cdot 3, 2 \cdot 0) = (6,0) \not\sim (3,6) = (1 \cdot 3 + 2 \cdot 0, 2 \cdot 3).$$

Theorem 2.4 *The two fields $\mathbf{Y}$ and $\mathbf{C}$ are isomorphic.*

Proof. Indeed, consider the mapping f from $\mathbf{Y}$ to $\mathbf{C}$:

$$f : x = \frac{a}{b} \mapsto \begin{cases} ab^{-1} & \left(\frac{a}{b} \neq 0_Y\right) \\ 0 & \left(\frac{a}{b} = 0_Y\right). \end{cases}$$

Then, we can see easily the followings: 1)$f(x+y) = f(x)+f(y)$, 2)$f(x \cdot y) = f(x)f(y)$, 3)$f(1_Y) = 1$, and 4)f is a one to one and onto mapping from $\mathbf{Y}$ to $\mathbf{C}$.

We define a unary operation φ_Y on $\mathbf{Y}$ as

$$\varphi_Y\left(\frac{a}{b}\right) = \frac{b}{a}.$$

For the inverse element of $x = \frac{a}{b} \neq 0_Y$, we shall denote it by x^{-1}.

Definition 2.2 *We define a binary operation $/$ on $\mathbf{Y}$ as follows: For any x, $y \in \mathbf{Y}$*

$$x/y = x \cdot \varphi_Y(y) = \begin{cases} xy^{-1} & (y \neq 0_Y), \\ 0 & (y = 0_Y). \end{cases}$$

We will call the field $\mathbf{Y}$ with the operation φ_Y *0-divisible field* or *the Yamada field.*

Theorem 2.5 $\mathbf{C}$ *becomes a 0-divisible field.*

Proof. Indeed, in $\mathbf{C}$, a unary operation $\varphi = f \circ \varphi_Y \circ f^{-1}$ is induced by the homomorphic f from the 0-divisible field $\mathbf{C}$. Then, for any $z \in \mathbf{C}$,

$$\varphi(z) = \begin{cases} z^{-1} & (z \neq 0)\,, \\ 0 & (z = 0)\,. \end{cases}$$

We, however, would like to state that *the division by zero* $z/0 = 0$ *is essentially, just the definition, and we can derive all properties of the division by zero, essentially, from the definition.* Furthermore, by the idea of this session, we can introduce the fundamental concept of the divisions (fractions) in any field.

We should use the 0-divisible field $\mathbf{Y}$ for the complex numbers field $\mathbf{C}$ as complex numbers, by this simple modification.

The above introduction of the Yamada field is natural and very interesting itself. Meanwhile, H. Okumura [71] found that for the introduction of fractions a/b in the Yamada field, it is enough with the simple definition that for $b \neq 0$

$$\frac{a}{b} = ab^{-1}$$

and for $b = 0$

$$\frac{a}{b} = ab.$$

Therefore, indeed, we can say that the construction of fields containing division by zero was very simple.

2.4 By the intuitive meaning of fractions (division) by H. Michiwaki

We will introduce an another approach. The division b/a may be defined **independently of the product**. Indeed, in Japan, the division b/a ; b **waru** a (**jozan**) is defined as how many a exists in b, this idea comes from subtraction a repeatedly. (Meanwhile, product comes from addition). In Japanese language for "division", there exists such a concept independently of product. H. Michiwaki and his 6 years old daughter Eko Michiwaki said for the result $100/0 = 0$ that the result is clear, from the meaning of fractions independently of the concept of product and they said: $100/0 = 0$ does not mean that $100 = 0 \times 0$. Meanwhile, many mathematicians had a confusion for the result. Her understanding is reasonable and may be acceptable. $100/2 = 50$ will mean that we divide 100 by 2, then each will have 50. $100/10 = 10$ will mean that we divide 100 by 10, then each will have 10. $100/0 = 0$ will mean that we do not divide 100, and then nobody will have at all and so 0. Furthermore, they said then the rest is 100; that is, mathematically;

$$100 = 0 \cdot 0 + 100.$$

Now, all mathematicians may accept the division by zero $100/0 = 0$ with natural feelings as a trivial one.

For simplicity, we shall consider the numbers on non-negative real numbers. We wish to define the division (or fraction) b/a following the usual procedure for its calculation, however, we have to take care for the division by zero. As the first principle, for example, for $100/2$ we shall consider it as follows:

$$100 - 2 - 2 - 2 - ... - 2.$$

How many times can we subtract 2? At this case, it is 50 times and so, the fraction is 50. As the second case, for example, for $3/2$ we shall consider it as follows:

$$3 - 2 = 1$$

and the rest (remainder) is 1, and for the rest 1, we multiple 10, then we consider similarly as follows:

$$10 - 2 - 2 - 2 - 2 - 2 = 0.$$

Therefore $10/2 = 5$ and so we define as follows:

$$\frac{3}{2} = 1 + 0.5 = 1.5.$$

By these procedures, for $a \neq 0$ we can define the fraction b/a. Here we do not need the concept of product. Except the zero division, all results for fractions are valid and accepted. Now, we shall consider the zero division, for example, $100/0$. Since

$$100 - 0 = 100,$$

that is, by the subtraction $100 - 0$, 100 does not decrease. Then, we can not say that we were able to subtract any from 100. Therefore, the subtract number should be understood as zero; that is,

$$\frac{100}{0} = 0.$$

We can understand this as follows. Division by 0 means that it does not divide 100 and so, the result is 0. Similarly, we can see that

$$\frac{0}{0} = 0.$$

As a conclusion, we should define the zero division as, for any b

$$\frac{b}{0} = 0.$$

For complex numbers, we can consider the division $\frac{z_1}{z_2}$, similarly, by using the Euler formula

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\}$$

for $|z_j| = r_j$ and $\arg z_j = \theta_j$. The problem may be reduced to one of the division $\frac{r_1}{r_2}$.

H. Michiwaki checked this subsection and recalled his documents as follows (2018.1.8.0:43): $\exp(0) = 0, 1$ (H. Michiwaki: 2016.3.21.), $0^0 = 0$ (H. Michiwaki : 2014.9.21, 2015.11.7, 2016.2.14.), $\cos 0 = 0, 1$ (H. Michiwaki: 2016.3.16.), $aF_a = bF_b$ (H. Michiwaki: 2015.11.17.), $\omega = v/r$ (H. Michiwaki: 2014.2.28.).

See [51] for the details.

As some sense of the division by zero, we note the following fact:

We will consider the division of the plane by n lines in the general position. Then, the number of the formed domains are given by the formula

$$F(n) = \frac{1}{2}(n^2 + n + 2).$$

For $n = 0$, we have $F(0) = 1$; that means that we do not divide the plane.

Similarly, we will consider the division $F(n)$ of the space $\mathbf{R^3}$ by n planes with general positions, then we have:

$$F(1) = 2$$

and

$$F(n + 1) = F(n) + \frac{1}{2}(n^2 + n + 2).$$

Therefore, for $n = 0$,

$$F(1) = F(0) + 1$$

and therefore,

$$F(0) = 1;$$

that is, 0-division means that we do not divide.

2.5 Other introduction of general fractions

By the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$.

By considering the values of functions with mean values of functions, we can introduce the general fractions. Note here that the Cauchy integral formula may be considered as a mean value theorem. The mean values will be considered as a fundamental concept in analysis. – In the concept of the division by zero calculus, we will refer to the exact meaning of this sentence.

On the division by zero in our theory, we will need essentially (not exact sense, we will state later some exact sense) only one new assumption in our mathematics that for the elementary function $W = 1/z$, $W(0) = 0$. However, for algebraic calculation of the division by zero, we have to follow the law of the Yamada field. For functions, however, we have to consider the concept of **the division by zero calculus**, as we will develop the details later with many applications.

We stated, on the division by zero, the importance of the definition of the division by zero $z/0$. However, we note that in our definition it is given as a **generalization** or **extension** of the usual fraction. Therefore, we will not be able to give its precise meaning at all. For this sense, we do not know the direct meaning of the division by zero. It looks **like a black hole**. In order to know its meaning, we have to examine many properties of the division by zero by applications.

However, we will purse some more direct meanings for the division by zero.

2.6 Ankur Tiwari's great discovery of the division by zero $1/0 = \tan(\pi/2) = 0$ on 2011

We got an important information on the Ankur Tiwari's great discovery of the division by zero $1/0 = \tan(\pi/2) = 0$ on 2011. Since the information was not known for us and among many colleagues, we would like to state our opinions on his great discovery.

We found the surprising information on the publications ([120, 121]) of Ankur Tiwari on the division by zero at 2020/02/25 : 9 : 45 as follows:

http://www.ankurtiwari.in/downloads/bnrf-paper.pdf

and

https://www.bookdepository.com/
Andhakar-Autobiography-Ankur-Tiwari/9788192373515?

2.6.1 Tiwari's basic ideas

We can understand Tiwari's basic ideas from the 7 pages paper, precisely.

Since the division by zero $z/0$ is not possible in the usual sense that $z/0 = X$ and $z = 0 \times X$ are the same, we have to consider some definition of the division by zero $z/0$.

His first idea: for the fraction

$$B = \frac{A}{Q},$$

we will consider it as follows: it is from the general form

$$A = B \times Q + R.$$

Therefore, for $Q = 0$, we have

$$A = R,$$

and he considers that the division by zero $z/0$ is zero and the remainder is z. This great idea comes from Mahavira (about 800 - about 870).

For his great idea, we have to refer to the same idea and the exact proof that our colleague Hiroshi Michiwaki had, on our early stage discovery of the division by zero (23 Feb. 2014).

His second idea is follows:

For a value of a function $F(z)$, he considers that

$$F(z) = \lim_{\delta \to 0} \frac{F(z-\delta) + F(z+\delta)}{2};$$

that is, with the mean value. And he obtained the very important results

$$\frac{1}{0} = 0, \quad \tan\frac{\pi}{2} = 0,$$

from the functions $y = 1/x$ and $y = \tan x$, respectively.

Of course, we considered the same way on our initial stage of our discovery of the division by zero.

However, with his idea, we will not be able to derive the important result, for example, for the function

$$f(x) = \frac{1}{x^2},$$

$f(0) = 0$.

Furthermore, in his definition, when do not exist the limits, he will not be able to give the definition.

2.6.2 Conclusion

Incidentally, when we find his publications, we are writing the Annoucement 549; an answer for the question whether mathematics is innovation (creation) or discovery. There we stated that mathematics is the real existence and not innovation. Mathematics exists independently of our existence, independently of time and energy. We have to say that mathematics was created by God. – Absolute existences. Indeed, we wrote: What is mathematics?

No.81, May 2012(pdf 432kb)
www.jams.or.jp/kaiho/kaiho-81.pdf

in Japanese, in details with human beings.

In particular, mathematics is over logic, we consider so.

From these ideas, we would like to say that the division by zero was discovered by Ankur Tiwari on 2011 based on his 7 pages article at this moment.

One basic reason is that he got the great ideas on the great history of India on the problem:

Brahmagupta (598 - 670),

Mahavira (about 800 - about 870),

and

Bhaskara II (1114 - 1185).

The second important reason is on his estimation for the results obtained; he admits the importance of the results in a highly way as we see from the document of 7 pages.

Therefore, we had sent the e-mail to him as follows:

Dear Ankur Tiwari:

Indeed, you are great and your discovery is very important. Since my English ability is poor, I first wrote the attached Announcement 550 for its importance in Japanese.

The main points are:

You are the first man of discovery of the division by zero,

Your passion and high estimation to the discovery are important factors.

I would like to send you; Congratulations!!!
You will be extremely happy with the great discovery.
We thought so.
I would like to write a new version as in

viXra:1903.0184 submitted on 2019-03-10 20:57:02, Who Did Derive First the Division by Zero 1/0 and the Division by Zero Calculus $\tan(\pi/2) = 0, \log 0 = 0$ as the Outputs of a Computer?

And I would like to add your important discovery in my book in details.

With best regards,
Sincerely yours,

Saburou Saitoh
2020.2.28.05:00

Now we think that any estimation ability is important; based on this idea, for the facts that CSEB and Chhattisgarh Academia gave the high estimation on his discovery we would like to express our great respects to them.

Meanwhile, for example, the division by zero is the generalized inverse - in the sense of **Moore-Penrose generalized inverse** - for the fundamental equation $aX = b$ and the inverse is fundamental and popular for the equation. Therefore, since our initial stage of the division by zero study, we stated repeatedly that the division by zero is trivial and clear all. However, over those 6 years, our world may not be accepted our opinion on its importance. Therefore, we are looking for its importance with many evidences over 1100 items.

In addition, we would like to refer to our paper ([18]) that will contain the division by zero as a very special case.

2.6.3 Misha Gromov defined that $\frac{0}{0} = 0$

At 2020.2.29.08 : 00, we obtained the e-mail from José Manuel Rodríguez Caballero:

Dear Saitoh,
Look at page 5 of the following paper (0 / 0 = 0)
https://www.ihes.fr/ gromov/wp-content/uploads/2018/08/structre-serch-entropy-july5-2012.pdf

José M.

Surprisingly enough, in the article ([39]) Misha Gromov **defined** that

$$\frac{0}{0} = 0$$

on June 25, 2013.

2.6.4 Could Brahmagupta derive the result $1/0 = 0$ from his result $0/0 = 0$?

Tiwari considers that the result $1/0 = 0$ is derived from the result $0/0 = 0$ as in

$$\frac{1}{0} = \frac{1}{\frac{0}{0}} = \frac{1 \times 0}{0} = \frac{0}{0} = 0.$$

This curious logic may not be accepted and contrary, we think that Brahmagupta was not, in general, able to consider the division by zero $1/0 = 0$. Look ([72]) for this opinion.

2.7 W. Hövel's interpretation in integers

W. Hövel gave the pleasant interpretation:

> **Dividing integer Numbers:**
>
> A mother invites kids to dinner. She cooks beans. She has M beans in her pot. Now she wants to share the beans fairly among the kids. Her math is very natural; she can only count. So she goes around the table and always gives the K kids sitting at the table a bean on their plate. She repeats this until all of the beans are distributed. Now it can happen that some children have one bean less than the other. That's unfair! So she gathers the excess beans back into her pot, which will contain m beans after the

division. Now everyone is satisfied and you can draw up a balance sheet:

M : number of beans in the mother's pot before division

m : number of beans in the mother's pot after division

K : number of kids

k : number of beans on the kid's plate after division

$M = k \times K + m$

Special case $M < K$:

There are more kids at the table than beans in the pot. To be fair, the mother has to collect all the beans back into their pot. The kids were given nothing to eat.

$$m = M$$

$$k = 0$$

Special case $K = 0$:

There are no kids at the table. After the division procedure, the mother still has $m = M$ beans in her pot, just as in the case of $M < K$ above. She sees no difference between these two cases, the pot is still full. Thus $k = 0$, the kids were given nothing to eat.

This is the famous problem that SABUROU SAITOH solved.

Special case $M \gg 1, K \ll M$:

Many beans were cooked in mother's pot and the kids were given a large number of beans on their plates. The beans look more and more like a bean soup. It looks like continuous . Private note for SABUROU SAITOH by Wolfhard Hövel (2020.10.9.17:10).

In particular, note that in the framework of integers, he can give the definition of m/n in the good and natural way.

We note that the fractions $m//n$ in the sense of Wolfhard Hövel in the integers are given by

$$m//n = \begin{cases} \left[\frac{m}{n}\right] & (m, n \quad \text{same sign}) \\ \left\{\frac{m}{n}\right\} & (m, n \quad \text{different signs}) \\ 0 & (\text{for} \quad n = 0). \end{cases}$$

Here, $[x]$ is floor(x) and $\{x\}$ is ceiling(x).

3 STEREOGRAPHIC PROJECTIONS

For a great meaning and importance, we will see that the point at infinity is represented by zero. Of course, we saw that for the fundamental function $W = 1/z$, since $1/0 = 0$, we see that the point at infinity is represented by zero.

3.1 The point at infinity is represented by zero

By considering the stereographic projection, we will be able to see that the point at infinity is represented by zero.

Consider the sphere in the space (ξ, η, ζ) with its radius $1/2$ put on the complex $z = x + iy$ plane with its center $(0, 0, 1/2)$ as in $\xi = x, \eta = y$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the sphere onto the complex $z(= x + iy)$ plane; that is,

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}.$$

If $\zeta = 1$, then, by the division by zero, the north pole corresponds to the origin $(0, 0) = 0$.

Here, note that

$$x^2 + y^2 = \frac{\zeta}{1-\zeta}.$$

For $\zeta = 1$, by the division by zero, we should consider as $1/0 = 0$, not from the expansion

$$\frac{\zeta}{1-\zeta} = -1 - \frac{1}{\zeta - 1},$$

– the division by zero calculus that will be discussed in details later –.

We will consider the unit sphere $\{(\xi, \eta, \zeta); \xi^2 + \eta^2 + \zeta^2 = 1\}$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the sphere onto the (x, y) plane; that is,

$$(\xi, \eta, \zeta) =$$

$$\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-1/(x^2+y^2)}{1+1/(x^2+y^2)}\right).$$

Then, we see that the north pole corresponds to the origin.

Next, we will consider the semi-sphere (ξ, η, ζ) with its center $C(0,0,1)$ and its radius 1 on the origin on the (x, y) plane. From the center $C(0,0,1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the semi- sphere onto the complex (x, y) plane; that is,

$$x = \frac{\xi}{1-\zeta}, y = \frac{\eta}{1-\zeta}.$$

If $\zeta = 1$, then, by the division by zero, the center C corresponds to the origin $(0,0)$.

Meanwhile, we will consider the mapping from the open unit disc with its center at the origin onto $\mathbf{R}^2$ in one to one and onto

$$\xi = \frac{x\sqrt{x^2+y^2}}{1+x^2+y^2}, \quad \eta = \frac{y\sqrt{x^2+y^2}}{1+x^2+y^2}$$

or

$$x = \frac{\xi}{\sqrt{\rho(1-\rho)}}, y = \frac{\eta}{\sqrt{\rho(1-\rho)}}; \quad \rho^2 = \xi^2 + \eta^2.$$

Note that the point $(x, y) = (0,0)$ corresponds to $\rho = 0; (\xi, \eta) = (0,0)$.

Furthermore, with many examples we will show that the point at infinity is represented by zero geometrically and analytically, in the sequel. We have had to change our basic idea for our space since Euclid.

3.2 A contradiction of classical idea for $1/0 = \infty$

The infinity ∞ may be considered by the idea of the limiting, however, we had considered it as a number, for sometimes, typically, the point at infinity was represented by ∞ for some long years. However, then ∞ is not clear and is not a definite number, but some ideal and vague one. For this fact, we will show a formal contradiction.

We will consider the stereographic projection by means of the sphere with its radius 1/2

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

from the complex $z = x + iy$ plane onto the sphere. Then, we obtain the correspondences

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}$$

and

$$\xi = \frac{1}{2}\frac{z+\overline{z}}{z\overline{z}+1}, \eta = \frac{1}{2i}\frac{z-\overline{z}}{z\overline{z}+1}, \zeta = \frac{z\overline{z}}{z\overline{z}+1}.$$

In general, two points P and Q_1 on the diameter of the sphere correspond to z and z_1, respectively if and only if

$$z\overline{z_1} + 1 = 0. \tag{3.1}$$

Meanwhile, two points P and Q_2 on the symmetric points on the sphere with respect to the plane $\zeta = \frac{1}{2}$ correspond to z and z_2, respectively if and only if

$$z\overline{z_2} - 1 = 0. \tag{3.2}$$

If the point P is the north pole or the origin, then the points Q_1 and Q_2 are the same point. Then, the identities (3.1) and (3.2) are not valid that show a contradiction.

Meanwhile, if we write (3.1) and (3.2)

$$z = -\frac{1}{\overline{z_1}}$$

and

$$z = \frac{1}{\overline{z_2}},$$

respectively, we see that the division by zero (1.2) is valid.

3.3 Natural meanings of $1/0 = 0$

We can see our division by zero for many fractions. We will show the simple examples.

For constants a and b satisfying

$$\frac{1}{a} + \frac{1}{b} = k, \quad (\neq 0, \mathrm{const.})$$

the function

$$\frac{x}{a} + \frac{y}{b} = 1$$

passes the point $(1/k, 1/k)$. If $a = 0$, then, by the division by zero, $b = 1/k$ and $y = 1/k$; this result is natural.

We will consider the line $y = m(x-a) + b$ through a fixed point $(a, b); a, b > 0$ with its gradient m. We set $A(0, -am + b)$ and $B(a - (b/m), 0)$ that are common points with the line and both lines $x = 0$ and $y = 0$, respectively. Then,

$$\overline{AB}^2 = (-am + b)^2 + \left(a - \frac{b}{m}\right)^2.$$

If $m = 0$, then $A(0, b)$ and $B(a, 0)$, by the division by zero, and furthermore

$$\overline{AB}^2 = a^2 + b^2.$$

Then, the line AB is a corresponding line between the origin and the point (a, b). Note that this line has only one common point with both lines $x = 0$ and $y = 0$. Therefore, this result will be very natural in a sense. – Indeed, we can understand that the line $\overline{AB}$ is broken into two lines $(0, b) - (a, b)$ and $(a, b) - (a, 0)$, suddenly. Or, the line AB is one connecting the origin and the point (a, b).

The general line equation through fixed point (a, b) with its gradient m is given by

$$y = m(x - a) + b \tag{3.3}$$

or, for $m \neq 0$

$$\frac{y}{m} = x - a + \frac{b}{m}.$$

By $m = 0$, we obtain the equation $x = a$, by the division by zero. This equation may be considered as the cases $m = \infty$ and $m = -\infty$, and these cases may be considered by the strictly right logic with the division by zero.

By the division by zero, we can consider the equation (3.3) as a general line equation.

In the Lami's formula for three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfying

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0},$$

$$\frac{\|\mathbf{A}\|}{\sin\alpha} = \frac{\|\mathbf{B}\|}{\sin\beta} = \frac{\|\mathbf{C}\|}{\sin\gamma},$$

if $\alpha = 0$, then we obtain

$$\frac{\|\mathbf{A}\|}{0} = \frac{\|\mathbf{B}\|}{0} = \frac{\|\mathbf{C}\|}{0} = 0.$$

Here, of course, α is the angle of $\mathbf{B}$ and $\mathbf{C}$, β is the angle of $\mathbf{C}$ and $\mathbf{A}$, and γ is the angle of $\mathbf{A}$ and $\mathbf{B}$.

For the Newton's formula; that is, for a C^2 class function $y = f(x)$, the curvature K at the origin is given by

$$K = \lim_{x \to 0} \left| \frac{x^2}{2y} \right| = \left| \frac{1}{f''(0)} \right|,$$

we have for $f''(0) = 0$,

$$K = \frac{1}{0} = 0.$$

Recall the formula

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx = -\frac{2}{n},$$

for

$$n = \pm 1, \pm 2, ..., ...$$

Then, for $n = 0$, we have

$$b_0 = -\frac{2}{0} = 0.$$

Furthermore, we will see many examples in this book.

3.4 Double natures of the zero point $z = 0$

Any line on the complex plane arrives at the point at infinity and the point at infinity is represented by zero. That is, a line, indeed, contains the origin; the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it is represented by $z = 0$. Later, we will see this property by analytic geometry and the division by zero calculus in many situations.

However, for the general line equation

$$ax + by + c = 0,$$

by using the polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$r = \frac{-c}{a\cos\theta + b\sin\theta}.$$

When $a\cos\theta + b\sin\theta = 0$, by the division by zero, we have $r = 0$; that is, we can consider that the line contains the origin. We can consider so, in the natural sense. We can define so as a line with the compactification and the representation of the point at infinity - the ideal point.

For the envelop of the lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{3.4}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{3.5}$$

The origin of this parabolic function is excluded from the envelop of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis.

– This function may be considered as a function with zero gradient and passing the point at infinity; however, the point at infinity is represented by 0, the origin; that is, the line may be considered as the x axis. Furthermore, then we can consider the x axis as a tangential line of the parabolic function, because they are gradient zero at the point at infinity. –

Furthermore, we can say later that the x axis $y = 0$ and the parabolic function have the zero gradient at **the origin**; that is, in the reasonable sense the x axis is a tangential line of the parabolic function.

Indeed, we will see the surprising property that the gradient of the parabolic function at the origin is zero.

Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is a native tangential line of the parabolic function. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (3.5) for the line (3.4) is given by

$$\left(\frac{p}{m}, \frac{2p}{m}\right)$$

and it is $(0,0)$ for $m = 0$.

We can see that the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is the reflected one of the point at infinity.

3.5 Puha's horn torus model

V. V. Puha discovered the mapping of the extended complex plane to a beautiful horn torus at (2018.6.4.7:22) and its inverse at (2018.6.18.22:18).

Incidentally, independently of the division by zero, Wolfgang W. Däumler has various special great ideas on horn torus as we see from his site:

Horn Torus & Physics (https://www.horntorus.com/) Geometry Of Everything, intellectual game to reveal engrams of dimensional thinking and proposal for a different approach to physical questions ...

Indeed, W. W. Däumler was presumably the first (1996) who came to the idea of the possibility of a mapping of extended complex plane onto the horn torus. He expressed this idea on his private website (http://www.dorntorus.de). He was also, apparently, the first to point out that zero and infinity are represented by one and the same point on the horn torus model of extended complex plane.

We will consider the three circles stated by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2, \tag{3.6}$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with its $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

$$\eta = \frac{2y\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

and

$$\zeta = \frac{(x^2 + y^2 - 1)\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center $(0, 0, 1/2)$ and the horn torus. This is the desired point on the horn torus for the point on the plane.

Indeed, we denote tentatively a point with (ξ_1, η_1, ζ_1) on the horn torus. Then, we have, from the relation between a point (ξ, η, ζ) on the Riemann sphere and the correspondent point (ξ_1, η_1, ζ_1) on the horn torus

$$\xi_1 = \xi \frac{\zeta_1 - 1/2}{\zeta - 1/2}, \quad \eta_1 = \eta \frac{\zeta_1 - 1/2}{\zeta - 1/2}.$$

We set

$$\sqrt{\xi^2 + \eta^2} = t, \quad \zeta_1 - \frac{1}{2} = n, \quad \zeta - \frac{1}{2} = m.$$

Then, since the point (ξ_1, η_1, ζ_1) is on the horn torus, from the identity

$$n^2 + \left(n\frac{t}{m} - \frac{1}{4}\right)^2 = \frac{1}{16},$$

we obtain the identity

$$n = \frac{mt}{2(m^2 + t^2)}.$$

Therefore, we obtain

$$\xi_1 = \frac{\xi}{2} \frac{\sqrt{\xi^2 + \eta^2}}{(\zeta - 1/2)^2 + \xi^2 + \eta^2},$$

$$\eta_1 = \frac{\eta}{2} \frac{\sqrt{\xi^2 + \eta^2}}{(\zeta - 1/2)^2 + \xi^2 + \eta^2}$$

and

$$\zeta_1 = \frac{1}{2} \frac{(\zeta - 1/2)\sqrt{\xi^2 + \eta^2}}{(\zeta - 1/2)^2 + \xi^2 + \eta^2} + \frac{1}{2}.$$

Hence, in terms of (x, y), we have the desired results.

The inversion is given by

$$x = \xi \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 - \zeta + \frac{1}{2}\right)^{(-1/2)} \tag{3.7}$$

and

$$y = \eta \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 - \zeta + \frac{1}{2}\right)^{(-1/2)}. \tag{3.8}$$

In these formulas, we can see the division by zero

$$\frac{1}{0} = 0,$$

naturally that shows the mapping of the point $(0, 0, 1/2)$ to $(0, 0)$.

At first, the model shows the strong symmetry of the domains $\{|z| < 1\}$ and $\{|z| > 1\}$ and they correspond to the lower part and the upper part of the horn torus, respectively. The unit circle $\{|z| = 1\}$ corresponds to the circle

$$\xi^2 + \eta^2 = \left(\frac{1}{2}\right)^2, \quad \zeta = \frac{1}{2}$$

in one to one way. Of course, the origin and the point at infinity are the same point and correspond to $(0, 0, 1/2)$. Furthermore, the inversion relation

$$z \longleftrightarrow \frac{1}{\overline{z}}$$

with respect to the unit circle $\{|z| = 1\}$ corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (\xi, \eta, 1 - \zeta)$$

and similarly,

$$z \longleftrightarrow -z$$

corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (-\xi, -\eta, \zeta)$$

and

$$z \longleftrightarrow -\frac{1}{\overline{z}}$$

corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (-\xi, -\eta, 1 - \zeta)$$

(H.G.W. Begehr: 2018.6.18.19:20).

However, we can see directly the important negative properties that the mapping is isogonal (equiangular) and infinitely small circles correspond to infinitely small circles are not valid, as in analytic functions. Of course, circles to circles mapping property is, in general, not valid as in the case of the stereographic projection mapping.

We note that only zero and numbers a of the form $|a| = 1$ have the property : $|a|^b = |a|, b \neq 0$. Here, note that we can also consider $0^b = 0$ ([54]). The symmetry of the horn torus model agrees perfectly with this fact. Only zero and numbers a of the form $|a| = 1$ correspond to points on the plane described by equation $\zeta - 1/2 = 0$. Only zero and numbers a of the form $|a| = 1$ correspond to points whose tangent lines to the surface of the horn torus are parallel to the axis ζ.

3.6 Däumler's horn torus model

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model (2018.8.18.09):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

Our situation is invariant by rotation around ζ axis, and so we shall consider the problem on the ξ, ζ plane.

Let $N(0, 0, 1)$ be the north pole. Let $P'(\xi, \eta, \zeta)$ denote a point on the Riemann sphere and let $z = x + iy$ be the common point with the line NP' and $\zeta = 0$ plane (: $z = x + iy$); that is P' is the stereographic projection map of the point $z = x + iy$ onto the unit sphere.

Let $M(1/4, 0, 1/2)$ be the center of the circle (3.6). Let P'' be the common point of the line $SP'(S = S(0, 0, 1/2))$ and the circle (3.6).

Let Q' be $(0, 0, \zeta)$ that is the line $Q'P'$ is parallel to the x axis. Let Q'' and M'' be the common points with the ζ axis and $\xi = 1/4$ with the parallel line to the x axis through the point P'', respectively.

Further, we set $\alpha = \angle OSP' = \angle P''IS = (1/2)\angle P''MS$ (I := I(1/2,0,1/2)). We set P for the point on the horn torus such that $\phi = \angle SMP$ and Q be the point on the ζ axis such that the line QP is parallel to the x axis.

Then, we have:

$$\overline{P'Q'} = \frac{1}{2}\sin\alpha,$$

$$\overline{P''M''} = \frac{1}{4}|\cos(2\alpha)|,$$

$$\overline{P''Q''} = \frac{1}{4}(1-\cos(2\alpha)),$$

the length of latitude through P' is

$$2\pi\overline{P'Q'} = \pi\sin\alpha,$$

and the length of latitude through P''

$$2\pi\overline{P''Q''} = \frac{\pi}{2}(1-\cos(2\alpha)) = \pi\sin^2\alpha.$$

Similarly, we have

$$2\pi\overline{QP} = \frac{\pi}{2}(1-\cos\phi).$$

In order to become the conformal mapping from the point P' to the point P, we have the identity

$$d\alpha : d\phi = \sin\alpha : 1-\cos\phi;$$

that is we have the differential equation

$$\frac{d\alpha}{\sin\alpha} = \frac{d\phi}{1-\cos\phi}.$$

Note here that the radius of the circle (3.6) is half of the stereographic projection mapping circle (the Riemann sphere). We solve this differential equation as, with an integral constant C

$$\log|\tan\frac{\alpha}{2}| = -\cot\frac{\phi}{2} + C.$$

For this derivation of the differential equation, see the detail comments in the site : **conformal mapping sphere $\leftrightarrow$ horn**

torus with beautiful figures and many informations, by W. W. Däumler. In order to check his idea, we will give a complete proof analytically in the last part of this session.

Using the correspondence

$$\alpha = 0 \leftrightarrow \phi = 0,$$

or

$$\alpha = \pi/2 \leftrightarrow \phi = \pi$$

or

$$\alpha = \pi \leftrightarrow \phi = 2\pi,$$

we have $C = 0$. Note that $\tan(\pi/2) = 0$, $\cot(\pi/2) = 0$ and $\log 0 = 0$ ([54]). Note also that the function $y = e^x$ takes two values 1 and 0 at $x = 0$. We will see them in Sections 5, 8 and 11.

Therefore,

$$\phi = 2\cot^{-1}(-\log \mid \tan(\alpha/2) \mid) \tag{3.9}$$

or

$$\alpha = 2\tan^{-1}(e^{(-\cot(\phi/2))}). \tag{3.10}$$

Next, note that

$$\tan\frac{\alpha}{2} = |z|$$

and

$$\alpha = 2\tan^{-1}|z|. \tag{3.11}$$

We thus have

$$\phi = 2\cot^{-1}(-\log|z|) \tag{3.12}$$

and the inverse is

$$|z| = e^{-\cot(\phi/2)}. \tag{3.13}$$

We thus obtain the complicated conformal mapping for the z plane to the horn torus by (3.12) and (3.10). The inverse conformal mapping for the horn torus to the complex z plane is given by (3.9) and (3.13).

For the integral constant C, Däumler considers the general constant C and stated that:

> I don't recognize a big problem with constant C. What are the crucial points? As I stated, all mappings from sphere to horn torus and inverse with any real C are conformal, but only the mappings with $C = 0$ are bijective. Respectively with
>
> $$\alpha = 2\tan^{-1}(p \cdot |z|)$$
>
> and
>
> $$|z| = \frac{\tan(\alpha/2)}{p},$$
>
> all mappings from complex plane to sphere and inverse with real $p > 0$ are conformal, but bijective only when $p = 1$, what is the normal Riemannian stereographic projection. Main thing is to have at least one solution ($C = 0$) in this topic, and we can keep other constants, C not equal 0 and p not equal 1, for special cases in different context.

For this very interesting topics, see his site.

We can represent the direct Däumler mapping from the z plane onto the horn torus as follows (V. V. Puha: 2018.8.28.22:31):

With (3.12),

$$\xi = \frac{x \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2 + y^2}}, \tag{3.14}$$

$$\eta = \frac{y \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2 + y^2}}, \tag{3.15}$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}. \tag{3.16}$$

Indeed, at first, we have

$$SP := L = 2 \cdot \frac{1}{4}\sin\frac{\phi}{2} = \frac{1}{2}\sin\frac{\phi}{2}, \tag{3.17}$$

$$\sqrt{\xi^2+\eta^2} = L\cos\left(\frac{\pi}{2}-\frac{\phi}{2}\right) = L\sin\frac{\phi}{2},$$

and

$$\sqrt{\xi^2+\eta^2} = \frac{1}{2}\sin^2\frac{\phi}{2}.$$

From the simple relations

$$\xi = \frac{x\sqrt{\xi^2+\eta^2}}{\sqrt{x^2+y^2}}, \eta = \frac{y\sqrt{\xi^2+\eta^2}}{\sqrt{x^2+y^2}}, \tag{3.18}$$

and

$$\zeta = -L\sin\left(\frac{\pi}{2}-\frac{\phi}{2}\right) + \frac{1}{2},$$

we have the desired representations.

We will give the inversion formula of the Däumler mapping. From (3.18) we have

$$x = \frac{\xi\sqrt{x^2+y^2}}{\sqrt{\xi^2+\eta^2}}, y = \frac{\eta\sqrt{x^2+y^2}}{\sqrt{\xi^2+\eta^2}}. \tag{3.19}$$

Hence, it is enough to represent $\sqrt{x^2+y^2}$ in terms of ξ, η, ζ on the horn torus. From (3.13), (3.17) and

$$T = \sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2},$$

we have the inversion formula from the horn torus to the x, y plane.

$$x = \frac{\xi}{\sqrt{\xi^2+\eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2+\eta^2+\zeta^2)}}{\sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2}} \right\} \tag{3.20}$$

and

$$y = \frac{\eta}{\sqrt{\xi^2+\eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2+\eta^2+\zeta^2)}}{\sqrt{\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2}} \right\}. \tag{3.21}$$

Properties of the Däumler conformal mapping

The Däumler conformal mapping stated is very complicated, however, has very beautiful properties. We will see its elementary properties.

The circle $|z| = r$ is mapped to the circle:

$$\xi^2+\eta^2 = \frac{1}{4}\left\{\sin\frac{\phi}{2}\right\}^4, \quad \zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}$$

with

$$\frac{\phi}{2} = \cot^{-1}(-\log r).$$

In particular, note that the unit circle $r = 1$ is mapped to the circle

$$\xi^2+\eta^2 = \left(\frac{1}{2}\right)^2, \quad \zeta = \frac{1}{2}.$$

Here, note also that $\phi = \pi$, by using the division by zero calculus, from

$$\frac{1}{\tan(\phi/2)} = 0.$$

We have the relation

$$\frac{\eta}{\xi} = \frac{y}{x},$$

but for $y = mx$

$$\zeta = -\frac{1}{4}\sin\left\{2\cot^{-1}\left(-\frac{1}{2}(\log x^2 + \log(1+m^2))\right)\right\}.$$

Furthermore, the inversion relation

$$z \longleftrightarrow \frac{1}{\bar{z}}$$

with respect to the unit circle $\{|z| = 1\}$ corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (\xi, \eta, 1-\zeta)$$

and similarly,

$$z \longleftrightarrow -z$$

corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (-\xi, -\eta, \zeta)$$

and

$$z \longleftrightarrow -\frac{1}{\bar{z}}$$

corresponds to the relation

$$(\xi, \eta, \zeta) \longleftrightarrow (-\xi, -\eta, 1-\zeta).$$

Of course, the conformal mapping of Däumler is important, however, its mapping is very involved and the difference with the Puha mapping is just the shift on the circle of longitude and the Puha mapping is very simple. Furthermore the Puha mapping is clear in the geometrical correspondence. Therefore, we will be able to enjoy the Puha mapping for the horn torus model.

Proof of the Däumler conformal mapping

In order to confirm the Däumler comformal mapping and at the same time, in order to see its analytical structure, we will examine it. In this subsection, for simplicity we use L, N with $L = \log(x^2 + y^2)$ and $N = L/(4 + L^2)$.

First, we calculate the first order derivatives.

$$\frac{\partial \xi}{\partial x} = \frac{8y^2 - 8x^2 L + 2y^2 L^2}{(x^2 + y^2)^{3/2}(4 + L^2)^2},$$

$$\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = \frac{-2xy(2 + L)^2}{(x^2 + y^2)^{3/2}(4 + L^2)^2},$$

$$\frac{\partial \eta}{\partial y} = \frac{2(4x^2 - 4y^2 L + x^2 L^2)}{(x^2 + y^2)^{3/2}(4 + L^2)^2},$$

$$\frac{\partial \zeta}{\partial x} = \frac{-2x(-4 + L^2)}{(x^2 + y^2)(4 + L^2)^2}$$

and

$$\frac{\partial \zeta}{\partial y} = \frac{-2y(-4 + L^2)}{(x^2 + y^2)(4 + L^2)^2}.$$

Next, we wish to have the relation between

$$(d\sigma)^2 = (d\xi)^2 + (d\eta)^2 + (d\zeta)^2$$

and

$$(ds)^2 = (dx)^2 + (dy)^2.$$

From

$$d\xi = \frac{2(-xydy(2 + L)^2 + dx(4y^2 - 4x^2 L + y^2 L^2))}{(x^2 + y^2)^{3/2}(4 + L^2)^2},$$

$$d\eta = \frac{2(-xydx(2 + L)^2 + dy(4x^2 - 4y^2 L + x^2 L^2))}{(x^2 + y^2)^{3/2}(4 + L^2)^2},$$

and

$$d\zeta = \frac{-2(xdx + ydy)(-4 + L^2)}{(x^2 + y^2)(4 + L^2)^2},$$

we obtain the beautiful identity

$$(d\sigma)^2 = \frac{4(ds)^2}{(x^2+y^2)(4+L^2)^2}. \tag{3.22}$$

The next and final crucial point is the relation of

$$\frac{dx}{ds}, \frac{dy}{ds}$$

and

$$\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}, \frac{d\zeta}{d\sigma}.$$

This may be done directly by division by $d\sigma$ in (3.22). Indeed, we have:

$$\frac{d\xi}{d\sigma} = \frac{dx(y^2 - 4x^2N) + dy(-xy - 4xyN)}{ds(x^2+y^2)}, \tag{3.23}$$

$$\frac{d\eta}{d\sigma} = \frac{dx(-xy - 4xyN) + dy(x^2 - 4y^2N))}{ds(x^2+y^2)}, \tag{3.24}$$

and

$$\frac{d\zeta}{d\sigma} = \frac{-xdx(-4+L^2) - ydy(-4+L^2)}{ds\sqrt{x^2+y^2}(4+L^2)}. \tag{3.25}$$

On a point $P_0(\xi_0, \eta_0, \zeta_0)$ on the horn torus we consider two smooth curves passing the point

$$f_j(\xi, \eta, \zeta) = 0, \quad j = 1, 2.$$

At the point P_0, we denote the values of $\frac{d\xi}{d\sigma}, \frac{d\eta}{d\sigma}, \frac{d\zeta}{d\sigma}$ by λ_j, μ_j, ν_j, respectively. Then, for the angle Φ made by the curves at the point P_0 we have

$$\cos\Phi = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2. \tag{3.26}$$

The corresponding relations on the x, y plane are as follows:

For the corresponding curves on the x, y plane

$$g_j(x, y) = 0, \quad j = 1, 2,$$

at the corresponding point $Q_0(x_0, y_0)$, we denote the values of $\frac{dx}{ds}, \frac{dy}{ds}$ by α_j, β_j, respectively. Then, for the angle ϕ of the curves at the point $Q_0(x_0, y_0)$ we have

$$\cos\phi = \alpha_1\alpha_2 + \beta_1\beta_2. \tag{3.27}$$

We wish to prove that (3.26) = (3.27), by formal calculation.

Note that from (3.23), we have for $(x, y) = (x_0, y_0)$, here, for simplicity we shall use (x, y) at Q_0

$$\lambda_j = \frac{\alpha_j(y^2 - 4x^2N) + \beta_j(-xy - 4xyN)}{x^2 + y^2}.$$

Similarly, from (3.24),

$$\mu_j = \frac{\alpha_j(-xy - 4xyN) + \beta_j(x^2 - 4y^2N)}{x^2 + y^2},$$

and from (3.25),

$$\nu_j = \frac{-x\alpha_j(-4 + L^2) - y\beta_j(-4 + L^2)}{\sqrt{x^2 + y^2}(4 + L^2)}.$$

When we insert these in the right side of (3.26), we obtain the right side of (3.27). In this subsection, for these formal calculations, we used MATHEMATICA.

The sources of the last two subsections on the horn torus models are given by [28]. Professor Tsutomu Matsuura kindly checked the calculations by MATHEMATICA in this subsection independently of Okumura and gave numerical experiments with computer graphics that show the Däumler conformal mapping. Of course, Puha also gave such numerical experiments with beautiful figures.

3.7 Absolute function theory

We will discuss on Däumler's horn torus model from some fundamental viewpoints.

First of all, note that in the Puha mapping and the Däumler mapping, and even in the classical stereographic mapping, we find the division by zero $1/0 = 0/0 = 0$.

What is the number system?

What are the numbers? What is the number system? For these fundamental questions, we can say that the numbers are complex numbers $\mathbf{C}$ and the number system is given by the Yamada field with the simple structure **as a field containing the division by zero.**

Nowadays, we have still many opinions on these fundamental questions, however, this subsection excludes all those opinions as in the above.

What is the natural coordinates?

We represented the complex numbers $\mathbf{C}$ by the complex plane or by the points on the Riemann sphere. On the complex plane, the point at infinity is the ideal point and for the Riemann sphere representation, we have to accept the **strong discontinuity**. From these reasons, the numbers and the numbers system should be represented by the Däumler's horn torus model that is conformally equaivalent to the extended complex plane.

What is a function?, and what is the graph of a function?

A function may be considered as a mapping from a set of numbers into a set of numbers.

The numbers are represented by Däumler's horn torus model and so, we can consider that a function, in particular, an analytic function can be considered as a mapping from Däumler's horn torus model into Däumler's horn torus model.

Absolute function theory

Following the above considerings, for analytic functions when we consider them as the mappings from Däumler's horn torus model into Däumler's horn torus model we would like to say that it is an **absolute function theory.**

For the classical theory of analytic functions, discontinuity of functions at singular points will be the serious problems and the theory will be quite different from the new mathematics, when we consider the functions on the Däumler's horn torus model. Even for analytic function theory on bounded domains, when we consider their images on Däumler's horn torus model, the results will be very interesting.

New mathematics and future mathematicians

The structure of Däumler's horn torus model is very involved and so, we will need some computer systems like MATHEMATICA and Isabelle/HOL system for our research activity. Indeed, for the analytical proof of the conformal mapping of Däumler, we had to use MATHEMATICA, already. Here, we will be able see some future of mathematicans.

For the properties of horn torus with physical applications, see [7].

See also the site of Däumler for some deep ideas:

https://www.horntorus.com/rotations.html

3.8 The theory of relativity by Einstein

For an interesting space model and division by zero, see also [59]:

> **Abstract:** The theory of relativity by Einstein can be interpreted by using a stereographic projection of points on the two dimensional complex plane onto a

three-dimensional sphere. Especially, $1/0 = \infty$ is interpreted smoothly, which means that if an infinite point extends infinitely, the infinite point on the Riemann sphere overlaps on the north pole. However, is this not simply saying that "infinite action is infinite"? This motif is the trigger for thinking of anti-Riemann geometric execution circumstances using a converse/inverse-Riemann sphere. So, we define and use "converse", "inverse" and "anti" as logic. Then we use this logic to derive an anti-Einstein field with a new operation. As a result, when anti-0 $[0 \Rightarrow 1]$ is an emergence symbol, the Mitsuyoshi operator uses a function as a simple means of connecting quantum theory and relativity and we hypothesize that $(0 \equiv \infty) = 1$. As a result, when anti-0 $[0 \Rightarrow 1]$ is an emergence symbol, we are led to the hypothesis that $(0 = \infty = 1$ by using the Mitsuyoshi operator as a simple means of connecting quantum theory and relativity.

4 MIRROR IMAGE WITH RESPECT TO A CIRCLE

For simplicity, we will consider the unit circle $|z| = 1$ on the complex $z = x + iy$ plane. Then, we have the reflection formula

$$z^* = \frac{1}{\bar{z}} \tag{4.1}$$

for any point z, as is well-known ([2]). For the reflection point z^*, there is no problem for the points $z \neq 0, \infty$. As the classical result, the reflection of zero is the point at infinity and conversely, for the point at infinity we have the corresponding point as the zero point. The reflection is a one to one correspondence and onto mapping between the inside and the outside of the unit circle, by considering the point at infinity.

Are these correspondences, however, suitable? Does there exist the point at ∞, really? Is the point at infinity corresponding to the zero point, by the reflection? Is the point at ∞ reasonable from the practical point of view? Indeed, where can we find the point at infinity? Of course, we know and see pleasantly the point at infinity on the Riemann sphere, however, on the complex z-plane it seems that we can not find the corresponding point. When we approach the origin on a radial line on the complex z plane, it seems that the corresponding reflection points approach *the point at infinity* with the direction (of the radial line).

With the concept of the division by zero, there is no the point at infinity ∞ as numbers. For any point z such that $|z| > 1$, there exists the unique point z^* by (4.1). Meanwhile, for any point z such that $|z| < 1$ except $z = 0$, there exits the unique point z^* by (4.1). Here, note that for $z = 0$, by the division by zero, $z^* = 0$. Furthermore, we can see that

$$\lim_{z \to 0} z^* = \infty, \tag{4.2}$$

however, for $z = 0$ itself, by the division by zero, we have $z^* = 0$. This will mean a strong discontinuity of the functions $W = \frac{1}{z}$

and (4.1) at the origin $z = 0$; that is a typical property of the division by zero. This strong discontinuity may be looked in the above reflection property, physically.

The result is a surprising one in a sense; indeed, by considering the geometrical correspondence of the mirror image, we will consider the center corresponds to the point at infinity that is represented by the origin $z = 0$. This will show that the mirror image is not followed by this concept; the correspondence seems to come from the concept of one-to-one and onto mapping.

Should we exclude the point at infinity, from numbers? We were able to look the strong discontinuity of the division by zero in the reflection with respect to circles, physically (geometrical optics). The division by zero gives a one to one and onto mapping of the reflection (4.1) from the whole complex plane onto the whole complex plane.

The infinity ∞ may be considered as in (4.2) **as the usual sense of limits,** however, the infinity ∞ is not a definite number.

Meanwhile, we would like to refer to the following interesting fact:

In the classical book [5]

D. H. Armitage and S. J. Gardiner did not refer to the images of the centers of circles and bolls, not at all.

We consider a circle on the complex z plane with its center z_0 and its radius r. Then, the mirror image relation p and q with respect to the circle is given by

$$p = z_0 + \frac{r^2}{\overline{q - z_0}}.$$

For $q = z_0$, we have, by the division by zero,

$$p = z_0,$$

For a circle

$$Az\overline{z} + \overline{\beta}z + \beta\overline{z} + D = 0; \quad A > 0, D: \text{ real number},$$

or

$$\left(z + \frac{\beta}{A}\right)\overline{\left(z + \frac{\beta}{A}\right)} = \frac{|\beta|^2 - AD}{A^2},$$

the points z and z_1 are in the relation of the mirror images with respect to the circle if and only if

$$Az_1\overline{z} + \overline{\beta}z_1 + \beta\overline{z} + D = 0,$$

or

$$\overline{z_1} = -\frac{\overline{\beta}z + D}{Az + \beta}$$

$$= -\frac{\overline{\beta}}{A} - \frac{1}{A}\left(D - \frac{|\beta|^2}{A}\right)\frac{1}{z - \left(-\frac{\beta}{A}\right)}.$$

The center $-\beta/A$ corresponds to the center itself, as we see from the division by zero.

On the x, y plane, we shall consider the inversion relation with respect to the circle with its radius R and with its center at the origin:

$$x' = \frac{xR^2}{x^2 + y^2}, \quad y' = \frac{yR^2}{x^2 + y^2}.$$

Then, the line

$$ax + by + c = 0$$

is transformed to the line

$$R^2(ax' + by') + c((x')^2 + (y')^2) = 0.$$

In particular, for $c = 0$, the line $ax + by = 0$ is transformed to the line $ax' + by' = 0$. This correspondence is one-to-one and onto, and so the origin $(0, 0)$ has to correspond to the origin $(0, 0)$.

Furthermore, we will see many examples in this book.

For the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0$$

and for the similar correspondences

$$x' = \frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}, \quad y' = \frac{a^2 b^2 y^2}{b^2 x^2 + a^2 y^2},$$

the origin corresponds to itself.

The pole (x_1, y_1) of the line

$$ax + by + c = 0$$

with respect to a circle with its radius R and with its center (x_0, y_0) is given by

$$x_1 = x_0 - \frac{aR^2}{ax_0 + by_0 + c}$$

and

$$y_1 = y_0 - \frac{bR^2}{ax_0 + by_0 + c}.$$

If $ax_0 + by_0 + c = 0$, then we have $(x_1, y_1) = (x_0, y_0)$.

Furthermore, for various higher dimensional cases the corresponding results are similar.

Anyhow, by the horn torus models of Puha and Däumler, we can see the whole situation of the reflection mappings or inversions clearly, because we can see the zero point and the point at infinity as the same one point.

5 DIVISION BY ZERO CALCULUS

As the number system containing the division by zero, the Yamada field structure is completed. Its structure is simple and natural, as we stated.

However for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of unique determination of the results and for some deep reasons. See [57].

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i},$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)},$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i},$$

when we enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot \left[\frac{1}{z+i}\right|_{z=-i} = (-2i) \cdot 0 = 0,$$

we have the different value.

In many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero to functions and we will need to check the results obtained, in some practical viewpoints. We will refer to this delicate problem with many examples.

The short version of this section was given by [96] in the Proceedings of the International Conference. See also [3]. However, the contents are mainly restricted to the differential equations for the conference topics.

5.1 Introduction of the division by zero calculus

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n \tag{5.1}$$

we will define

$$f(a) = C_0. \tag{5.2}$$

For the correspondence (5.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (5.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

Apart from the motivation, we define the division by zero calculus by (5.2)**.** With this assumption, we can obtain many new results and new concepts. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problem. – In this viewpoint, **the division by zero calculus may be considered as an axiom.**

In addition, we will refer to the naturality of the division by zero calculus.

Recall the Cauchy integral formula for an analytic function $f(z)$; for an analytic function $f(z)$ around $z = a$ and for a smooth simple Jordan closed curve γ enclosing one time the point a, we have

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

Even when the function $f(z)$ has any isolated singularity at the point a, we assume that this formula is valid as the division by zero calculus.

We **define** the value of the function $f(z)$ at the singular point $z = a$ with the above Cauchy integral.

For example, for $|z| = 1, z = x + iy$, the identity

$$\left(\frac{z+1}{z-1}\right)^2 = \frac{x+1}{x-1}$$

holds even for the case $z = 1$ by the division by zero calculus.

The identities

$$\frac{\exp(ix)+1}{\exp(ix)-1} = \frac{\cos x + 1}{i \sin x}$$

and

$$\left(\frac{\exp(ix)+1}{\exp(ix)-1}\right)^2 = \frac{\cos x + 1}{\cos x - 1}$$

valid even for the case $x = 0$ and their values at $x = 0$ are 0 and $2/3$, respectively, by the division by zero calculus, as we see from the expansions

$$\frac{-2i}{x} + \frac{ix}{6} + \frac{ix^3}{360} + \cdots$$

and

$$\frac{-4}{x^2} + \frac{2}{3} + \frac{-x^2}{60} + \cdots,$$

respectively.

However, note that

$$\frac{z+1}{z-1}|_{z=1} = 1$$

and

$$\left(\frac{z+1}{z-1}\right)^2 |_{z=1} = 1$$

for $z = 1$ by the division by zero calculus.

Axiom for the division by zero calculus

Of course, for an axiom, we wish to give some simple good representation, because we have several equivalent representations, in general. We consider the axiom of the division by zero calculus by its definition. However, how will be the following representation?:

Axiom of the division by zero calculus: *For any negative integer n and for the function, for any fixed a*

$$f_n(z) = (z-a)^n$$

we assume that

$$f_n(a) = 0.$$

Of course, if the equation $f_n(z) = (z-a)^n = 0$ has a solution, then the solution has to be a. Indeed, we have considered to solve an equation by extending our concept in our mathematics.

Elementary properties of division by zero calculus

For any analytic functions $f(z), g(z)$ on $\{\epsilon < |z-a| < R\}$, In general,

$$(fg)(a) \neq f(a)g(a),$$

$$\left(\frac{1}{f}\right)(a) \neq \frac{1}{f(a)},$$

$$(f(g))(a) \neq f(b)g(a), b = g(a).$$

Here, of course, we assume that the domain of f is containing the point $b = g(a)$ and f is analytic around $b = g(a)$.

Meanwhile, we obtain:

$$(\alpha f + \beta g)(a) = \alpha f(a) + \beta g(a),$$

$$(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a),$$

$$(fg)'(a) = (f'g)(a) + (fg')(a),$$

and

$$\left(\frac{g}{f}\right)'(a) = \frac{g'f - gf'}{f^2}(a).$$

Simple and convenient facts:

For an analytic function $f(z)$ on $R_a = \{z; 0 < |z| < a\}$, if $f(-z) = -f(z)$ on R_a, then

$$f(0) = 0$$

and if $f(-z) = f(z)$ on R_a, then

$$f'(0) = 0.$$

We will give typical and various examples.

For the typical function $(\sin x)/x$, we have

$$\frac{\sin x}{x}(0) = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, by the division by zero calculus, we have, for the function $(\sin x)/x$

$$\frac{\sin x}{x}(0) = 1,$$

that is more reasonable in analysis.

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, **for the results by the division by zero we should check the results, case by case.**

This does not imply any incompleteness of mathematics, that is why, for example, for the product $f(z)g(z)$ of two analytic functions $f(z)$ and $g(z)$, for the value of $f(z)g(z)$ at a singular point $z = a$, we can consider its value in the both senses; that is,

$$f(z)g(z)|_{z=a}$$

and

$$f(z)|_{z=a} \cdot g(z)|_{z=a}.$$

Those values are, in general, different, in the division by zero calculus.

For example, for the simple example for the line equation on the x, y plane

$$ax + by + c = 0$$

we have, formally

$$x + \frac{by + c}{a} = 0,$$

and so, by the division by zero, we have, for $a = 0$, the reasonable result

$$x = 0.$$

This case may be considered as the case of $a \to \infty$.

For the equation $y = mx$, from

$$\frac{y}{m} = x,$$

we have, by the division by zero, $x = 0$ for $m = 0$. This gives the case $m = \pm\infty$ of the gradient of the line. – This will mean that the equation $y = mx$ represents the general line through the origin containing the line $x = 0$ in this sense. – This method was applied in many cases, for example see also [84, 68].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$

For this case, we can consider that

$$\frac{ax + by}{c} + \frac{c}{c} = 0,$$

that is always valid. **In this sense, we can divide an equation by zero.**

In the formula

$$\left(-\frac{1}{a}\sqrt{1-2ax+a^2}\right)' = \frac{1}{\sqrt{1-2ax+a^2}},$$

by the division by zero calculus, we obtain directly, for $a = 0$

$$(x)' = 1.$$

From the identity

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2},$$

by using the expansion

$$\pi^2\left(\frac{1}{\pi z} + \frac{\pi z}{6} + \cdots\right)^2$$

$$= \pi^2\left(\frac{1}{\pi^2 z^2} + \frac{1}{3} + \cdots\right),$$

we have the identity, from the division by zero calculus, immediately, for $z = 0$

$$\zeta(2) = \frac{\pi^2}{6}.$$

Meanwhile, from the expansions

$$\cot z = \frac{1}{z} + 2z\sum_{k=1}^{\infty} \frac{1}{z - k^2\pi^2}$$

([1], page 75, 4.3.70) and

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^2 - \cdots$$

([1], page 75, 4.3.91), by the division by zero calculus, we have the same result.

For the case

$$g(\eta) = \frac{1}{\cosh^2 \eta} - \frac{\tanh \eta}{\eta^3}, \quad (\eta > 0)$$

at $\eta = 0$, we have,

$$-1 + \frac{1}{3} = \frac{-2}{3},$$

separately (see, S. Watanabe ([123]), page 9, (2.4)).

Meanwhile, note that for the function $f(z) = z + \frac{1}{z}$, $f(0) = 0$, however, for the function

$$f(z)^2 = z^2 + 2 + \frac{1}{z^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$

In the formula

$$\frac{x^{a+1}}{a+1}\left(\log x - \frac{1}{x+1}\right),$$

for $a = -1$, we have, by the division by zero calculus,

$$\frac{1}{2}\left(\log x\right).$$

For $\{\pi < \arg z < \pi\}$ and for any complex number a, we obtain the important identity, by the division by zero calculus

$$\frac{z^a}{a}|_{a=0} = \log z.$$

For the applications, see, for example, [1],

$$4.4.65, 4.4.67, 4.4.69, 4.4.71, 4.6.52, 4.6.54, 4.6.56, 4.6.58.$$

Indeed, in calculus, this formula give a great impact.

For an analytic function $f(z)$ around $z = a$ such that

$$f(z) = f'(a)(z-a) + \frac{1}{2}f''(a)(z-a)^2 + \cdots, \quad f'(a) \neq 0,$$

we obtain

$$\frac{1}{f(z)}|_{z=a} = \frac{-f''(a)}{2f'(a)^2}.$$

Furthermore, for an analytic function $g(z)$ around $z = a$, we have

$$\frac{g(z)}{f(z)}|_{z=a} = \frac{g'(a)}{f'(a)} - \frac{g(a)f''(a)}{2f'(a)^2}.$$

For example, for the integral formula

$$\int z^n (\log z)^m dz = \frac{z^{n+1}}{n+1}(\log z)^m$$

$$-\frac{m}{n+1}\int z^n (\log z)^{m-1} dz \quad (n \neq -1)$$

([1], page 69: 4.1.51), we can obtain the right formula for $n = -1$, by the division by zero calculus, immediately

$$\int \frac{1}{z}(\log z)^m dz = (\log z)^{m+1} - m\int \frac{1}{z}(\log z)^m dz.$$

In this case, this is missing there.

In the integral, for $ac - b^2 > 0$

$$I = \int \frac{dx}{ax^2 + bx + c} = \int \frac{adx}{(ax+b)^2 + (ac - b^2)}$$

$$= \frac{1}{\sqrt{ac-b^2}} \arctan \frac{ax+b}{\sqrt{ac-b^2}},$$

for $\xi = \sqrt{ac - b^2} = 0$, from

$$I = \frac{1}{\xi} \arctan \frac{ax+b}{\xi}$$

$$= \frac{1}{\xi}\left(\frac{\pi}{2} - \frac{\xi}{ax+b} + \frac{1}{3}\left(\frac{\xi}{ax+b}\right)^3 - \ldots\right),$$

we have the right result, by the division by zero calculus

$$I = -\frac{1}{ax+b}.$$

For the integral

$$\int \frac{dx}{a+bx^2} = \sqrt{\frac{1}{ab}} \arctan\left(\sqrt{\frac{b}{a}} x\right) + C,$$

for $a = 0$, by the division by zero calculus, we have

$$\frac{1}{b}\int \frac{dx}{x^2} = -\frac{1}{b}\frac{1}{x} + C.$$

Of course, for $b = 0$, we have

$$\frac{1}{a}\int dx = \frac{1}{a}x + C.$$

Meanwhile, for the formula

$$\int_0^\infty \frac{\cos x}{x^2+a^2} dx = \frac{\pi e^{-a}}{2a}, \quad (a > 0),$$

we obtain the identity

$$\int_0^\infty \frac{\cos x}{x^2} dx = -\frac{\pi}{2}.$$

However, we have to consider the integral in a generalized sense, for example, in the distribution theory.

For the general mean formula, for $a_k > 0$,

$$M(t) = \left(\frac{1}{n}\sum_{k=1}^{n} a_k^t\right)^{(1/t)},$$

for $t = 0$, we obtain the geometrical mean, by the division by zero calculus,

$$M(0) = (a_1 a_2 a_3 \cdots a_n)^{\frac{1}{n}}.$$

From the definition of the division by zero calculus, directly, we obtain, simply:

For the function

$$\frac{\exp(ax)}{f(a)}, \quad f(a) = 0$$

if $f(z)$ is analytic around $z = 0$ and $f'(a) = f''(a) = ... = f^{(m)}(a) = 0$ and $f^{(m+1)}(a) \neq 0$, by the division by zero calculus, we obtain the identity

$$\frac{x^{m+1}\exp(ax)}{f^{(m+1)}(a)}.$$

When $f(D)$ is an (polynomial) ordinary differential operator with $D = d/dx$ and with constant coefficients, in the ordinary differential equation

$$f(D)y = \exp(ax),$$

if $f'(a) = f''(a) = ... = f^{(m)}(a) = 0$ and $f^{(m+1)}(a) \neq 0$, then it gives a special solution.

The result may be looked like a generalization of l'Hôpital's rule.

For the function

$$\frac{ab}{(\sqrt{a} - \sqrt{b})^2}$$

that was introduced by H. Okumura for his geometric problem on (2021.1.4.), we have the Laurent expansion, by WolframAlpha

$$\frac{4a^3}{(b-a)^2} + \frac{6a^2}{b-a} + \frac{7a}{4} - \frac{1}{8}(b-a) + \cdots,$$

however, by setting $B = \sqrt{b}$ and by the division by zero calculus for B at $B = \sqrt{a}$ we have the reasonable value

$$\frac{ab}{(\sqrt{a} - \sqrt{b})^2}(b = a) = a.$$

Furthermore, see many examples in [57].

5.2 Division by zero calculus for differentiable functions

We will give the definition of the division by zero calculus for more general functions over analytic functions.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{5.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense.

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 represents impossibility.

We will give its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x\to a} \frac{F(x)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

The identity (5.3) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x\to a} \frac{f(x)}{x-a}$$

BUT

$$\frac{f(x)}{x-a}|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x-a$ of the function $f(x)$ at $x = a$ which is defined independently of the value $f(a)$. This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n = 0$ for the general formula.

In the expression (5.3), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x = a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

Of course, by our definition

$$\frac{f(x)}{x-a}|_{x=a} = \frac{f(x)-f(a)}{x-a}|_{x=a},$$

however, here $f(a)$ may be replaced by any constant. This fact looks like showing that the function

$$\frac{1}{x-a}$$

is zero at $x = a$. Of course, this result is directly derived from the definition.

When we apply the relation (5.3) in the elementary formulas for differentiable functions, we can imagine some deep results. For example, from the simple formula

$$(u+v)' = u' + v',$$

we have the result

$$\frac{u(x)+v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition.

In the following well-known formulas, we have some deep meanings on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u\prime}{u} + \frac{v\prime}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v \frac{u'}{u} \right).$$

For the second order derivatives, we have the familiar formulas:

$$(uv)'' = u''v + 2u'v' + uv'',$$

$$\frac{d^2 f(g(t))}{dt^2} = f''(g(t))g'(t) + f'(g(t))g''(t),$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3}$$

and

$$\frac{d^2 x}{dy^2} = -\frac{d^2 y}{dx^2} \left(\frac{dy}{dx}\right)^{-3}.$$

As a version of l'Hôpital's theorem, we obtain the following idea.

We assume that f and g are differentiable of n orders at $x = a$ and we assume that

$$g(a) = g'(a) = ... = g^{(n-1)}(a) = 0, g^{(n)}(a) \neq 0.$$

Then, we **define**

$$\frac{f(x)}{g(x)}|_{x=a} := \frac{f(x)/(x-a)^n|_{x=a}}{g(x)/(x-a)^n|_{x=a}} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

The function

$$\frac{f(x)}{g(x)}$$

is not defined at $x = a$ in the usual sense, because $g(a) = 0$. The denominator $g(x)$ has not to be zero. The initial non-vanishing value is $g^{(n)}(a)$. Therefore, in order to catch the value we consider

$$\frac{g(x)}{(x-a)^n}|_{x=a}.$$

Therefore, in a natural sense, we can obtain the above desired definition. We gave the natural reason. This is our interpretation for the l'Hôpital's theorem. We gave the definition of $\frac{f(x)}{g(x)}|_{x=a}$ in the case $g(a) = 0$.

In this idea, we obtain

$$\frac{1}{x}|_{x=0} = \frac{1'}{x'}|_{x=0} = \frac{0}{1} = 0.$$

The representation of the higher order differential coefficients $f^{(n)}(a)$ is very simple and, for example, for the Taylor expansion we have the beautiful representation

$$f(a) = \sum_{n=0}^{\infty} \frac{f(x)}{(x-a)^n}|_{x=a} \cdot (x-a)^n.$$

Further

$$\frac{f(x)}{(x-a)^2}|_{x=a} = \frac{f''(a)}{2}$$

$$= \lim_{x\to 0} \frac{f(a+x) + f(a-x) - 2f(a)}{2x^2}.$$

Of course, our definition of

$$\frac{f(x)}{g(x)}|_{x=a}$$

and the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

are, in general, different.

For example,

$$\frac{1+(x-1)\log x}{\sin^2(x-1)}|_{x=1} = 1,$$

but

$$\lim_{x\to 1} \frac{1+(x-1)\log x}{\sin^2(x-1)} = \infty.$$

Furthermore, note that

$$\lim_{x\to 0} \frac{x+2x^2\sin(1/x)}{x} = 1,$$

but

$$\frac{x+2x^2\sin(1/x)}{x}|_{x=0}$$

is not defined in our new sense. However, with the division by zero calculus for analytic functions, we have the value 3.

However, for some complicated statements of l'Hôpital's theorem, we can apply our definition and method for many cases in order to derive the same results. In our method it is necessary to consider only the point at $x = a$, meanwhile, in the l'Hôpital's theorem, we consider the behavior of the function on a neighborhood of the point $x = a$ except the point $x = a$. They are different concepts.

For the function

$$f(x) = x \sin\frac{1}{x},$$

if $f(0) = 0$, then the function is continuous at $x = 0$, however, it is not differentiable at the origin. By the division by zero calculus, we have, automatically

$$f(0) = 1.$$

The function

$$\frac{f(x)}{x}$$

is not defined at $x = 0$, however, we have

$$\frac{f(x)}{x}|_{x=0} = 0,$$

in the sense of the division by zero calculus.

The function

$$g(x) = x^2 \sin \frac{1}{x}$$

is defined at the point $x = 0$ as $g(0) = 0$ in the usual sense and by the division by zero calculus. We have

$$g'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

and is not defined at $x = 0$. However, with the division by zero calculus,

$$g'(0) = 1.$$

For an analytic function $W = f(z)$ on a domain D, of course, we have

$$W - f(z) = 0, \quad z \in D.$$

Then, note that for any $a \in D$ and for any integer n,

$$\frac{W - f(z)}{(z-a)^n} = 0, \quad z \in D.$$

Meanwhile, we have an interesting formula whose proof is simple:

Theorem 5.1. *Consider a family of absolutely continuous functions* $f_a(x)$ *that is analytic in* $a \in \mathbb{R} \setminus \{a_0\}$. *Let* $g_a(x) = f'_a(x)$ *and we assume that it is extensible to the point at* $a = a_0$ *as an absolutely continuous function, then*

$$f_{a_0}(x) = \int g_{a_0}(x)\, dx.$$

We will show examples:

1. Let $f_n(x) = \frac{(ax+b)^{n+1}}{a(n+1)}$ where $a \in \mathbb{R} \setminus \{0\}$ and $n+1$ is a positive integer. Then $g_n(x) = (ax+b)^n$ and

$$\left[\frac{(ax+b)^{n+1}}{a\,(n+1)}\right]_{n=-1} = \int (ax+b)^{-1}\,dx = \frac{\ln|ax+b|}{a}, \ a \neq 0;$$

by the same way we have

$$\left[\frac{(ax+b)^{n+1}}{a\,(n+1)}\right]_{a=0} = \int b^n dx = b^n x.$$

2. Let $f(x) = \frac{\arctan(x/a)}{a}$ where $a \in \mathbb{R} \setminus \{0\}$. In this case we get $g_n(x) = \frac{1}{x^2+a^2}$ and consequently

$$\left[\frac{\arctan(x/a)}{a}\right]_{a=0} = \int \frac{1}{x^2}dx = -\frac{1}{x}.$$

3. Let $f(x) = \frac{a^x}{\log a}$, $a > 0$. Then, we obtain

$$\left[\frac{a^x}{\log a}\right]_{a=1} = \int dx = x.$$

In this example, note that the function $f(x)$ may not be considered in the sense of the Laurent expansion in a. However, by setting $\log a = A$, we can obtain that:

$$\frac{e^{Ax}}{A}|_{A=0} = x,$$

by the division by zero calculus. In the formula

$$\int a^x dx = \frac{a^x}{\log a} + C,$$

for $a = 1$, the formula

$$\int 1^x dx = \frac{1^x}{\log 1} + C$$

is not valid.

Meanwhile, we obtain that

$$\left(\frac{1}{\log x}\right)_{x=1} = 0.$$

Indeed, we consider the function $y = \exp(1/x), x \in \mathbf{R}$ and its inverse function $y = \frac{1}{\log x}$. By the symmetric property of two functions with respect to the function $y = x$, we have the desired result.

Here, note that for the function $\frac{1}{\log x}$, we can not use the Laurent expansion around $x = 1$, and therefore, the result is not trivial.

In particular, note that the function $W = \exp(1/z)$ takes the **Picard's exceptional value** 1 at the origin $z = 0$, by the division by zero calculus.

Meanwhile, for the identity

$$\frac{a-b}{\log a - \log b},$$

for $a = b$, we should consider it in the following way. By substituting $\log a = A$ and $\log b = B$, from

$$\frac{\exp A - \exp B}{A - B},$$

by the division by zero calculus, we have the reasonable result for $A = B$,

$$\exp A = a.$$

This example shows that the concept of division by H. Michiwaki is not suitable for functions.

However, substitution methods are very delicate. For example, for the function

$$w = \frac{1+it}{1-it},$$

for $t = -i$, by the division by zero calculus, we have a good value $w = -1$. However, from the representation $z = e^{i\alpha}$ we have

$$\frac{1+z}{1-z} = i \cot \frac{\alpha}{2}$$

and for $\alpha = 0$ and $z = 1$, we have the contradiction $-1 = 0$. By considering the way

$$\frac{1+e^{i\alpha}}{1-e^{i\alpha}}$$

and when we consider it by the division by zero calculus in connection with α for $\alpha = 0$, we have the right value 0.

Theorem 5.2 *Consider a family of absolutely continuous functions $F_a(x)$ where $a \in I \subset R$, I is an open interval, and*

$$f_a(x) = \int F_a(x)dx.$$

If a point a_0 is a pole of order n of the analytic functions $f_a(x)$ as functions in a and there exists an analytic function $g : I \to \mathbb{R}$ for any fixed x such that $g(a,x) = (a-a_0)^n f_a(x)$ then

$$\int F_{a_0}(x)dx = \frac{g^{(n)}(a_0,x)}{n!}.$$

Proof. Using the Taylor theorem, we have, for any fixed x

$$g(a,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a_0,x)}{k!}(a-a_0)^k,$$

and by the division by zero calculus, we have

$$\int F_{a_0}(x)dx = f_{a_0}(x) = \left[\frac{1}{(a-a_0)^n} g\left(a, x\right)\right]_{a=a_0} = \frac{g^{(n)}\left(a_0, x\right)}{n!}.$$

Theorems 5.1 and 5.2 were discovered by S. Pinelas (see [96]).

We shall give examples.

1. For the integral

$$\int x(x^2+1)^a dx = \frac{(x^2+1)^{a+1}}{2(a+1)} \quad (a \neq -1),$$

we obtain, by the division by zero,

$$\int x(x^2+1)^{-1} dx = \frac{\log(x^2+1)}{2}.$$

2. For the integral

$$\int \sin ax \cos x dx = \frac{\sin ax \sin x + a \cos ax \cos x}{1-a^2} \quad (a^2 \neq 1),$$

we obtain, by the division by zero, for the case $a = 1$

$$\int \sin x \cos x dx = \frac{\sin^2 x}{2} - \frac{1}{4}.$$

3. For the integral

$$\int \sin^{\alpha-1} x \cos(\alpha+1)x dx = \frac{1}{\alpha} \sin^{\alpha} x \cos \alpha x,$$

we obtain, by the division by zero, for the case $\alpha = 0$

$$\int \sin^{-1} x \cos x dx = \log \sin x.$$

4. For the integral
$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx + b \cos bx),$$
for example, we can consider the case $a = bi$, by the division by zero calculus, and we can obtain the expected good result.

We can obtain many and many such identities.

We will state the formal theorem whose proof is trivial:

Theorem 5.3 *Consider an operator L that transforms functions $f_z(t)$ on a set T with analytic parameter z of an isolated singular point a into functions $L[f_z(t)] = F_z(s)$ on a set S. Assume that for the Laurent expansions around a point $a \in D$, a disc on the complex z plane with its center a, for any fixed t*
$$f_z(t) = \sum_{n=-\infty}^{\infty} f_n(t)(z-a)^n,$$
$$L[f_z(t)] = \sum_{n=-\infty}^{\infty} L[f_n(t)](z-a)^n.$$
Then we have
$$F_a(s) = L[f_a(t)].$$

We illustrate this result with examples:

1. Let $f_\lambda(t) = \frac{\sin(\lambda t)}{\lambda}$, where $\lambda \in \mathbb{R} \setminus \{0\}$. The Laplace transform of $f_\lambda(t)$ is
$$L\left[\frac{\sin(\lambda t)}{\lambda}\right] = \frac{1}{s^2 + \lambda^2}$$
for $\lambda \neq 0$. Then we have the identity
$$L[t] = \frac{1}{s^2}.$$

2. Let $f_{\mu,\lambda}(t) = \frac{e^{\mu t} - e^{\lambda t}}{\mu - \lambda}$, where $\mu \neq \lambda$. The Laplace transform of $f_{\mu,\lambda}(t)$ is

$$L\left[\frac{e^{\mu t} - e^{\lambda t}}{\mu - \lambda}\right] = \frac{1}{(s-\mu)(s-\lambda)}$$

for $\mu \neq \lambda$. Then we have the formula

$$L\left[te^{\lambda t}\right] = \frac{1}{(s-\lambda)^2}.$$

3. We consider the function

$$f(t) = \begin{cases} 2t, & \text{if} \quad 0 \leq t < 1; \\ 3 - t, & \text{if} \quad 1 \leq t < 2; \\ 0, & \text{if} \quad t \geq 2, \end{cases}$$

whose Laplace transform is

$$F(s) = \frac{1 - 2e^{-s} + e^{-3s}}{s^2} \qquad (s > 0)$$

([110]). Then, by l'Hôpital's rule, we can not derive the value at $s = 0$ as $7/2$, which is derived by the division by zero calculus.

4. For many generating functions we can obtain some interesting identities. For example, we will consider the mapping

$$\zeta \in C \setminus \{0\} \to F(z,\zeta) := \exp \frac{z}{2}\left(\zeta - \frac{1}{\zeta}\right).$$

Then, from

$$F(z,\zeta) = \sum_{n=-\infty}^{+\infty} J_n(z)\zeta^n,$$

we obtain the formula

$$F(z,0) = J_0(z).$$

5.3 Representations of the division by zero calculus by means of mean values

Here, we will give simple and pleasant introductions of the division by zero calculus by means of mean values that give an essence of the division by zero. In particular, we will introduce a new mean value for real valued functions in connection with the Sato hyperfunction theory.

In particular, we would like to say for this subsection that:

GOD loves mean values.

Complex variable functions case

For a function $f(z), z = x + iy$, we would like to see and realize the value $f(z)$ by

$$f(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta,$$

if the right hand side exists.

With this definition, we see immediately that the above fundamental properties are satisfied.

Of course, this mean value is very classical. However, note that we can introduce our division by zero calculus with this definition for some general complex variable functions with singularities.

Real variable functions case

For a real variable function $f(x)$, we would like to see and realize the value $f(x)$ by the mean value

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2} \left\{ f(x + \varepsilon) + f(x - \varepsilon) \right\},$$

if we can consider the right hand side.

With this definition, we will see the above fundamental case

$$f_n(x) = \frac{1}{x^n}.$$

Then, we have for odd n:

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2} \left\{ \frac{1}{(x+\varepsilon)^n} + \frac{1}{(x-\varepsilon)^n} \right\}.$$

$$= \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{x^n} & \text{for } x \neq 0. \end{cases}$$

This gives a good result for odd n.

However, for even n, the limit in the definition does not exist. However, for $x = 0$, from

$$\frac{1}{2} \left\{ \frac{1}{(x+\varepsilon)^n} + \frac{1}{(x-\varepsilon)^n} \right\}$$

$$= \frac{1}{\varepsilon^n},$$

if we consider the division by zero calculus for $\varepsilon = 0$, we have the desired result.

Therefore, we can give the representation (definition) as follows:

When there exists the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2} \{f(x+\varepsilon) + f(x-\varepsilon)\}$$

or

$$\frac{1}{2} \{f(x+\varepsilon) + f(x-\varepsilon)\}$$

has the meaning for $\varepsilon = 0$ with the division by zero calculus, we define the value $f(x)$ with the limit and with the value in the sense of division by zero calculus, respectively.

We can obtain the following theorem, easily

Theorem:

(1) If $f(x)$ is differentiable at $x = 0$, then

$$\frac{f(x)}{x}|_{x=0} = f'(0).$$

(2) If n is even and $f(x)$ is odd with respect to $x = 0$, then

$$\frac{f(x)}{x^n}|_{x=0} = 0.$$

(3) If $n, n \geq 3$ is odd and $f(x)$ is even , then

$$\frac{f(x)}{x^n}|_{x=0} = 0.$$

(4) If $f(x)$ is real analytic around at $x = 0$, with the division by zero calculus,

$$\frac{f(x)}{x^n}|_{x=0} = \frac{f^{(n)}(0)}{n!}.$$

As we show the example of $y = \log x$ and for other reasons, we will introduce the new mean value by changing ε by $i\varepsilon$ as follows:

When there exists the limit

$$\lim_{\varepsilon \to 0} \frac{1}{2} \{f(x + i\varepsilon) + f(x - i\varepsilon)\}$$

or

$$\frac{1}{2} \{f(x + i\varepsilon) + f(x - i\varepsilon)\}$$

has the meaning for $\varepsilon = 0$ with the division by zero calculus, we define the value $f(x)$ with the limit and with the value in the sense of division by zero calculus, respectively.

Of course, in this definition we assume that the complex variable functions $f(x + i\varepsilon)$ and $f(x - i\varepsilon)$ are given around the real line.

Examples:

1. For

$$f(x) = \frac{|x|}{x^n}, \quad n \geq 1,$$

we have

$$f(0) = 0.$$

For

$$f(z) = \frac{|z|}{z^n}, \quad n \geq 1,$$

we have

$$f(0) = 0.$$

2. For some functions that are not odd, we can obtain some reasonable values.

   For

$$f(x) = \log x,$$

   we have

$$f(0) = 0.$$

   Here, we consider the principal values on the domain $-\pi < \arg z < \pi$. Note that in this case, with the usual mean, we can not consider the value $\log 0$, but, by the complex mean value, we can consider the natural value $\log 0 = 0$ with the sense of the division by zero calculus.

   For $\log 0 = 0$, see also [54].

3. For

$$f(x) = \frac{a^x}{x}, \quad a > 0$$

   we have

$$f(0) = \log a.$$

4. For the function

$$f(x) = e^{\frac{1}{x}},$$

   we see that the limit in the definition does not exist, however, from the identity, for $x = 0$

$$\frac{1}{2}\left\{f(x + i\varepsilon) + f(x - i\varepsilon)\right\} = \cos\frac{1}{\varepsilon},$$

we have $f(0) = 1$ with the sense of the division by zero calculus. With the real mean value, we have the similar result.

For the complex valued function

$$f(z) = e^{\frac{1}{z}},$$

we have that

$$f(0) = 1$$

with the division by zero calculus.

5. For the function

$$f(x) = x \sin \frac{1}{x}$$

we have $f(0) = 0$ with the mean value method, however, by the division by zero calculus, we have $f(0) = 1$.

6. For the function

$$f(x) = \frac{e^{(1/x)}}{e^{(1/x)} - e^{(-1/x)}}$$

we have

$$\lim_{x \to +0} f(x) = 1$$

and

$$\lim_{x \to -0} f(x) = 0.$$

However, by the division by zero calculus and by the mean value method, we have $f(0) = 1/2$.

7. For the limit:

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2} \left\{ \frac{1}{(x + \varepsilon i)^n} + \frac{1}{(x - \varepsilon i)^n} \right\},$$

for the case $\varepsilon = y, z = x + iy$, it is represented as

$$\lim_{y \to 0} \Re \frac{1}{z^n}.$$

Then, with $z = re^{i\theta}$, we have

$$\Re \frac{1}{z^n} = \frac{1}{r^n} \Re e^{-in\theta} = \frac{1}{r^n} \cos n\theta.$$

Therefore, for

$$\theta = \pm \frac{\pi}{2n},$$

it is zero. On the line

$$\arg z = \pm \frac{\pi}{2n}$$

it is always zero. Then, we see that $x = x(y)$ and when y tends to zero, $x = x(y)$ also tends to zero. Indeed, we have the relations

$$y = \tan \left(\pm \frac{\pi}{2n} \right) x$$

and

$$\sum_m {}_nC_{2m} (-1)^m y^{2m} x^{n-2m} = 0.$$

Therefore, in this sense, the complex mean value tends to zero at the origin for all n.

From the viewpoint of the Sato hyperfunction theory

The form in the representation of real variable function may be looked as in the Sato hyperfunction, indeed as in the definition of the finite part of Hadamard (page 11, [48]). Therefore, we would like to consider the elementary relation of real valued functions, the division by zero and the Sato hyperfunction theory.

For an analytic function $f(z)$ around $z = x$, we will consider the Cauchy integral representation. However, following the basic idea of Sato, we will consider it as in the following way. Let γ_- be an analytic curve in the lower complex plane whose start point is $x - E$ and ends at the point $x + E$ $(E > 0)$. Let γ_+ be an analytic curve in the upper complex plane whose start point is $x - E$ and ends at the point $x + E$. Here, we are considering that both curves γ_- and γ_+ are near to the real line and they tends to the real line in some sense. Of course they are in some neighborhood of analytic domain of the function $f(z)$. Then, the Cauchy integral representation is as follows:

$$f(x) = \int_{\gamma_+} -\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - x}d\zeta - \int_{\gamma_-} -\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - x}d\zeta$$

$$= \int_{-E}^{E} \left(-\frac{1}{2\pi i}\right)\left(\frac{1}{\xi - x + i0} - \frac{1}{\xi - x - i0}\right) f(\xi)d\xi.$$

By setting

$$\delta(\xi - x) = \left(-\frac{1}{2\pi i}\right)\left(\frac{1}{\xi - x + i0} - \frac{1}{\xi - x - i0}\right),$$

we have the identity

$$f(x) = \int_{-E}^{E} \delta(\xi - x) f(\xi)d\xi.$$

This idea is the basic idea of the Sato hyperfunction theory. For an analytic function $f(z)$ we can consider one representation of the division by zero calculus, because we used the Cauchy integral representation. The serious problem is on the above identities and their interpretation. The curves γ_- and γ_+ approach to the real line and its result may be represented as in the above. In particular, the convergence is an essential problem in the Sato hyperfunction theory. Indeed, for some general functions, we can consider such limits. See the elementary facts, for example, [48, 49].

In order to see the notation, we will write

$$\delta(x) = \left(-\frac{1}{2\pi i}\right)\left(\frac{1}{x+i0} - \frac{1}{x-i0}\right)$$
$$= \left[-\frac{1}{2\pi i}\frac{1}{z}\right]$$

and the function

$$-\frac{1}{2\pi i}\frac{1}{z}$$

is a defining function of the generalized function $\delta(x)$.

In general, for a meromorphic function $F(z)$ around a part of the real line, we define a finite part of the function $F(x)$ in the sense of Hadamard by

$$f.p.F(x) = \frac{1}{2}(F(x+i0) + F(x-i0)).$$

Then, we see that

$$f.p.\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x+i0} + \frac{1}{x-i0}\right).$$

In this case, we found interestingly that

$$\frac{1}{x} = \lim_{\varepsilon\to 0}\frac{1}{2}\left(\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}\right).$$

Since this representation shows the zero property of the function $y = 1/x$ at the origin, we see that the very interesting property of singular points of analytic functions and the theory of the Sato hyperfunction theory.

In particular, note the important property that for a continuous function $f(x)$ except for an isolated point, its defining analytic function $F(z)$ may be represented by the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_R f(\xi)\frac{1}{\xi - z}d\xi,$$

if there exists the integral.

In general, apparently, we can not connect with the finite part of Hadamard and the mean value. In order to see some details, we will consider the prototype case of the function $1/x^2$.

We have the identity

$$\frac{1}{x^2}|_{x=0} = \int_{\gamma_+} -\frac{1}{2\pi i}\frac{1}{\zeta^3}d\zeta - \int_{\gamma_-} -\frac{1}{2\pi i}\frac{1}{\zeta^3}d\zeta$$
$$= \frac{1}{4\pi i}\left[\frac{1}{\zeta^2}\right]_{\gamma_+} - \frac{1}{4\pi i}\left[\frac{1}{\zeta^2}\right]_{\gamma_-}.$$

This can be considered as a mean value of the function $1/x^2$ around the origin. However, we can not connect with the mean value as in the real variable function and also, a finite part of Hadamard.

As a typical example in A. Kaneko ([48], page 11) in the theory of hyperfunction theory we see that for non-integers λ, we have

$$x_+^\lambda = \left[\frac{-(-z)^\lambda}{2i\sin\pi\lambda}\right] = \frac{1}{2i\sin\pi\lambda}\{(-x+i0)^\lambda - (-x-i0)^\lambda\}$$

where the left hand side is a Sato hyperfunction and the middle term is the representative analytic function whose meaning is given by the last term. For an integer n, Kaneko derived that

$$x_+^n = \left[-\frac{z^n}{2\pi i}\log(-z)\right],$$

where log is a principal value on $\{-\pi < \arg z < +\pi\}$. Kaneko stated there that by taking a finite part (C_0) of the Laurent expansion, the formula is derived. Indeed, we have the expansion, around an integer n,

$$\frac{-(-z)^\lambda}{2i\sin\pi\lambda}$$

$$= \frac{-z^n}{2\pi i}\frac{1}{\lambda - n} - \frac{z^n}{2\pi i}\log(-z)$$

$$- \left(\frac{\log^2(-z)z^n}{2\pi i \cdot 2!} + \frac{\pi z^n}{2i \cdot 3!}\right)(\lambda - n) + ...$$

([48], page 220).

By the division by zero calculus, however, we can derive this result from the Laurent expansion, immediately.

Meanwhile, M. Morimoto derived this result by using the Gamma function with the elementary means in [60], pages 60-62.

In addition, note that

$$\frac{\delta(x)}{x}|_{x=0} = -\delta'(x)|_{x=0} = 0$$

with

$$\delta(0) = 0.$$

In general,

$$x \cdot \delta^{(k)}(x) = -k\delta^{(k-1)}(x).$$

Here, however, for $k = 0$,

$$x \cdot \delta(x) = 0$$

(([48], page 12).

In general, we have, for $\alpha_j \geq \beta_j$

$$x^{(\alpha)} \cdot \delta^{(\beta)}(x) = (-1)^{\alpha)}\frac{\beta!}{(\beta - \alpha)!}\delta^{(\beta-\alpha)}(x),$$

for otherwise, we have 0 ([48], page 95 and page 216).

5.4 Difficulty in Maple for specialization problems

For the Fourier coefficients a_n

$$a_n = \int t\cos n\pi t dt = \frac{\cos n\pi t}{n^2\pi^2} + \frac{t}{n\pi}\cos n\pi t,$$

we obtain, by the division by zero calculus,

$$a_0 = \frac{t^2}{2}.$$

Similarly, for the Fourier coefficients a_n

$$a_n = \int t^2 \cos n\pi t dt = \frac{2t}{\pi^2 n^2} \cos n\pi t - \frac{2}{n^3\pi^3} \sin n\pi t + \frac{t^2}{n\pi} \sin n\pi t,$$

we obtain

$$a_0 = \frac{t^3}{3}.$$

For the Fourier coefficients a_k of a function

$$\frac{a_k \pi k^3}{4}$$

$$= \sin(\pi k)\cos(\pi k) + 2k^2\pi^2 \sin(\pi k)\cos(\pi k) + 2\pi(\cos(\pi k))^2 - \pi k,$$

for $k = 0$, we obtain, by the division by zero calculus, immediately

$$a_0 = \frac{8}{3}\pi^2$$

(see [128], (3.4)).

We have many such examples.

5.5 Reproducing kernels

We consider a reproducing kernel Hilbert space $H_K(E)$ on a set E. Then, for the reproducing property that for any $f \in H_K(E)$ and for any $p \in E$, $(f(\cdot), K(\cdot .p))_{H_K(E)} = f(p)$, we obtain the fundamental inequality

$$|f(p)| \leq \|f\|_{H_K(E)} \sqrt{K(p,p)}.$$

We set the normalized reproducing kernel $e_a(p)$ at a point a as norm 1 as

$$e_a(p) = \frac{K(p,a)}{\sqrt{K(a,a)}},$$

for the non-trivial case of $K(a,a) \neq 0$. If $K(a,a) = 0$, then for any function $f \in H_K(E)$, we have $f(a) = 0$ and $K(p,a) = \overline{K(a,p)} = 0$ for any point p. So, we have the identity

$$0 = \frac{0}{0}.$$

The function

$$K_{a,b}(x,y) = \frac{1}{2ab} \exp\left(-\frac{b}{a}|x-y|\right)$$

is the reprodiucing kernel for the space $H_{K_{a,b}}$ equipped with the norm

$$\|f\|^2_{H_{K_{a,b}}} = \int (a^2 f'(x)^2 + b^2 f(x)^2) dx$$

([110], pages 15-16). If $b = 0$, then, by the division by zero calculus

$$K_{a,0}(x,y) = -\frac{1}{2a^2}|x-y|$$

and this is the reproducing kernel for the space $H_{K_{a,0}}$ equipped with the norm

$$\|f\|^2_{H_{K_{a,b}}} = a^2 \int (f'(x)^2 dx.$$

Meanwhile, if $a = 0, K_{0,b}(x,y) = 0$, then it is the trivial reproducing kernel for the zero function space.

We denote by $\mathcal{O}(\{0\})$ the set of all analytic functions defined on a neighborhood of the origin.

Let $\{C_j\}_{j=0}^{\infty}$ be a positive sequence such that

$$\limsup_{j\to\infty} \sqrt[j]{C_j} < \infty.$$

Set

$$R \equiv \left(\limsup_{j\to\infty} \sqrt[j]{C_j}\right)^{-1} > 0$$

and define a kernel K by

$$K(z,u) \equiv \sum_{j=0}^{\infty} C_j\, z^j\, \overline{u}^j \quad (|z|, |u| < \sqrt{R}).$$

Then we have

$$H_K(\Delta(\sqrt{R})) = \left\{ f \in \mathcal{O}(\Delta(\sqrt{R})) \;:\; \sqrt{\sum_{j=0}^{\infty} \frac{|f^{(j)}(0)|^2}{(j!)^2 C_j}} < \infty \right\}$$

and the norm is given by the formula

$$\|f\|_{H_K(\Delta(\sqrt{R}))} = \sqrt{\sum_{j=0}^{\infty} \frac{|f^{(j)}(0)|^2}{(j!)^2 C_j}}$$

that is the reproducing kernel Hilbert space admitting the kernel ([110], page 35).

If some constant C_{j_0} is zero, then there is no problem, by considering that in the above statement

$$\frac{|f^{(j_0)}(0)|^2}{(j!)^2 C_{j_0}} = 0.$$

5.6 Ratio

On the real x line, we fix two different points $P_1(x_1)$ and $P_2(x_2)$ and we will consider the point, with a real number r

$$P(x;r) = \frac{x_1 + r x_2}{1 + r}.$$

If $r = 1$, then the point $P(x;1)$ is the mid point of two points P_1 and P_2 and for $r > 0$, the point P is on the interval (x_1, x_2). Meanwhile, for $-1 < r < 0$, the point P is on $(-\infty, x_1)$ and for $r < -1$, the point P is on $(x_2, +\infty)$. Of course, for $r = 0$, $P = P_1$. We see that when r tends to $+\infty$ and $-\infty$, P tends to the point P_2. We see the pleasant fact that by the division by zero calculus, $P(x, -1) = P_2$. For this fact we see that for all real numbers r correspond to all real line points.

In particular, we see that in many text books at the undergraduate course the formula is stated as a parameter representation of the line through two pints P_1 and P_2. However, if we do not consider the case $r = -1$ by the division by zero calculus, the classical statement is not right, because the point P_2 can not be considered.

On this setting, we will consider another representation

$$P(x; m, n) = \frac{mx_2 - nx_1}{m - n}$$

for the exterior division point $P(x; m, n)$ in $m : n$ for the points P_1 and P_2. For $m = n$, we obtain, by the division by zero calculus, $P(x; m, m) = x_2$. Imagine that the point $P(x; m, m) = P_2$ and the point P_2 seems to be the point P_1. Such a strong discontinuity happens for many cases. See also [57, 83].

For fixed two vectors $OA = a$ and $OB = b$ $(a \neq b)$, we consider two vectors $OA' = a' = \lambda a$ and $OB' = b' = \mu b$ with parameters λ and μ. Then, the common point x of the two lines AB and $A'B'$ is represented by

$$x = \frac{\lambda(1-\mu)a + \mu(\lambda - 1)b}{\lambda - \mu}.$$

For $\lambda = \mu$, we should have $x = 0$, by the division by zero. However, by the division by zero calculus, we have the curious result

$$x = (1 - \mu)a + \mu b.$$

By the division by zero, we can introduce the ratio for any complex numbers a, b, c as

$$\frac{AC}{CB} = \frac{c - a}{b - c}.$$

We will consider the **Appollonius circle** determined by the equation

$$\frac{AP}{PB} = \frac{|z-a|}{|b-z|} = \frac{m}{n} \tag{5.4}$$

for fixed $m, n \geq 0$. Then, we obtain the equation for the cirlce

$$\left| z - \frac{-n^2 a + m^2 b}{m^2 - n^2} \right|^2 = \frac{m^2 n^2}{(m^2 - n^2)^2} \cdot |b-a|^2. \tag{5.5}$$

If $m = n \neq 0$, the circle is the line in (5.5). For $|m| + |n| \neq 0$, if $m = 0$, then $z = a$ and if $n = 0$, then $z = b$. If $m = n = 0$ then z is a or b.

The representation (5.4) is valid always, however, (5.5) is not reasonable for $m = n$. The property of the division by zero depends on representations of formulas.

On the real line, the points $P(p), Q(1), R(r), S(-1)$ form a harmonic range of points if and only if

$$p = \frac{1}{r}.$$

If $r = 0$, then we have $p = 0$ that is now the representation of the point at infinity (H. Okumura: 2017.12.27.)

For two chords AB and CD of a fixed circle with a common point P in the inside of the circle, we have the relation

$$\frac{PA}{PC} = \frac{PD}{PB}.$$

If C=P=B, then we have

$$\frac{PA}{0} = \frac{PD}{0} = 0$$

and the formula is still valid.

5.7 Identities

For the decomposition of the linear fraction

$$W = \frac{az+b}{cz+d}$$

$$= \frac{a}{c} + \frac{bc-ad}{c(cz+d)}, \quad bc-ad \neq 0,$$

of course, the identity is valid for $c = 0$ by the division by zero calculus.

For example, we have the identity

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{(c-b)(a-c)(x-a)}$$

$$+ \frac{1}{(b-c)(b-a)(x-b)} + \frac{1}{(c-a)(c-b)(x-c)}.$$

By the division by zero calculus, the first term in the right hand side is zero for $x = a$, and

$$\frac{1}{(b-c)(b-a)(a-b)} + \frac{1}{(c-a)(c-b)(a-c)}.$$

This result is the same as

$$\frac{1}{(x-a)(x-b)(x-c)}\,(a)\,,$$

by the divison by zero calculus.

For the identity

$$\frac{1}{x(a+x)^2} = \frac{1}{a^2x} - \frac{1}{a(a+x)^2} - \frac{1}{a^2(a+x)},$$

we have the identity as $\frac{1}{x^3}$ for $a = 0$.

For the identity

$$f(z) = \Pi_{j=1}^{n}(z - z_j),$$

we have the identity

$$\left[\frac{f'(z)}{f(z)}\right]_{z=z_1}=\frac{1}{z_1-z_2}+...+\frac{1}{z_1-z_n}.$$

For the identity

$$\frac{mx+n}{ax^2+2bx+c}$$

$$=\frac{m}{2a}\frac{2ax+2b}{ax^2+2bx+c}+\frac{an-bm}{a}-\frac{1}{ax^2+2bx+c},$$

for $a=0$, we have

$$\frac{mx+n}{2bx+c}=\frac{x(bx+c)}{(2bx+c)^2}+\frac{2bnx+nc+bmx^2}{(2bx+c)^2}.$$

For the identity

$$I_n=(-1)^n n!\frac{1}{(a^2+x^2)^{(n+1)/2}}\sin(n+1)\theta$$

$$=\frac{(-1)^n n!}{2i}\left[\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right],z=x+iy=e^{i\theta},$$

we have, for $x=ai$

$$[I_n]_{x=ai}=\frac{(-1)^n n!}{2^{n+2}i^n}.$$

In the identity, for $-\pi\leq x\leq\pi$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2-a^2}=\frac{\pi\cos ax}{2a\sin a\pi}-\frac{1}{2a^2},$$

for $a=0$, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2}=\frac{1}{12}(\pi^2-3x^2).$$

In the identity, for $0 < x < 2\pi, |a| \leq 1$

$$\sum_{n=1}^{\infty} a^{2n-1} \frac{\sin[(2n-1)x]}{2n-1} = \frac{1}{2} \tan^{-1} \frac{2\sin x}{1-a^2},$$

for $a = 1$, we have, for $0 < x < \pi$,

$$\sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{2n-1} = \frac{\pi}{4}.$$

For the identities, for $0 \leq x \leq 2\pi$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} \cos(nx) = \frac{\pi}{2a\sinh(a\pi)} \cosh[a(\pi - x]] - \frac{1}{2a^2},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2-a^2} \cos(nx) = \frac{\pi}{2a\sin(a\pi)} \cos[a(\pi - x]] + \frac{1}{2a^2},$$

for $a = 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2).$$

In the identity

$$\frac{1}{x} - \frac{{}_nC_1}{x+1} + \frac{{}_nC_2}{x+2} + \cdots + (-1)^n \frac{{}_nC_n}{x+n}$$

$$= \frac{n!}{x(x+1)(x+2)\cdots(x+n)},$$

from the singular points, we obtain many identities, for example, from $x = 0$, we obtain the identity

$$-{}_nC_1 + \frac{{}_nC_2}{2} + \cdots + (-1)^n \frac{{}_nC_n}{n}$$

$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

We can derive many identities in this way.

For the rational equation

$$\frac{2(x-1)}{(x-1)(x+1)} = 1,$$

we obtain the natural solution $x = 1$ by the division by zero calculus. However, we did not consider so; that is, there is no solution for the equation.

For the equation

$$\frac{x - 4y + 2z}{x} = \frac{2x + 7y - 4z}{y} = \frac{4x + 10y - 6z}{z} = k,$$

from $k = 1$, we have the solution with parameter λ

$$x = y = \lambda, \quad z = 2\lambda.$$

We obtain also the natural solution

$$x = y = z = 0.$$

However, then $k = 0$.

For the equation

$$x - x = \frac{x}{x} \tag{5.6}$$

(Nathaniel Andika: 2019.6.22.05:36 in Quora), we have the solution $x = 0$.

On the history of mathematics, we have the nature that in order to solve equations, we extended the number system; for example, in order to solve the equation $x^2 = -1$, we introduced the complex numbers by introducing i. On this history, we can consider that in order to solve the fundamental equation (5.6) we introduced the division by zero $0/0 = 0$ by giving the meaning of $\frac{x}{x}$ at the point $x = 0$.

We consider the expansions

$$y = ax + bx^2 + cx^3 + \cdots \quad (a \neq 0)$$

and

$$x = Ay + By^2 + Cy^3 + \cdots,$$

then we have

$$A = \frac{1}{a},$$

$$B = -\frac{b}{a^3},$$

$$a^5 C = 2b^2 - ac,$$

and

$$a^7 D = 5abc - -a^2 d - 5b^3,$$

and so on.

If $a = 0$, then from

$$y = bx^2 + cx^2 + \cdots,$$

we have

$$bA^2 = 0, 2bAB = 0, \cdots$$

and so,

$$y \equiv 0$$

and

$$x \equiv 0.$$

Therefore, all the constants $a, b, c, \cdots; A, B, C, \cdots$ are zero. Hence,

$$A = \frac{1}{0} = 0$$

and

$$B = -\frac{0}{0} = 0.$$

G.J.O. Jameson ([45], page 426) notes the identity, for $n \geq 2, 1 \leq r \leq n$
$$\frac{r}{n} = \frac{r}{n}\frac{r-1}{n-1} + \frac{n-r}{n}\frac{r}{n-1}.$$
However, by the division by zero calculus, we see that this identity holds even the case $n = 1$ for any r.

Of course, the identity is also valid for $n = 0$ for any r with the division by zero calculus.

5.8 Inequalities

For the problem

$$f(x) = \frac{1}{(x-1)(x-2)} < 0,$$

we have the solution
$$1 < x < 2$$
in the usual sense. However, note that by the division by zero calculus
$$f(1) = -1$$
and
$$f(2) = -1.$$
Therefore, we have the solution

$$1 \leq x \leq 2.$$

Meanwhile, we know

Growth Lemma ([96], 267 page) *For the polynomial*

$$P(z) = a_0 + a_1 z + ... + a_n z^n (a_0, a_n \neq 0, n > 1)$$

we have the inequality with a sufficient r, for $|z| \geq r$

$$\frac{|a_n|}{2}|z|^n \leq |P(z)| \leq \frac{3|a_n|}{2}|z|^n.$$

At the point at infinity, since $P(z)$ takes the value a_0, the inequality is not valid more.

In the inequality

$$\pi < \frac{\sin \pi x}{x(1-x)} \leq 4 \quad (0 < x < 1)$$

([1], page 75, 4.3.82), the function takes π at $x = 0, 1$ and so we have the inequality

$$\pi \leq \frac{\sin \pi x}{x(1-x)} \leq 4 \quad (0 \leq x \leq 1).$$

Therefore, for inequalities, for the values of singular points by means of the division by zero calculus, we have to check the values, case by case.

5.9 We can divide the numbers and analytic functions by zero

In the division by zero like $1/0, 0/0$ the important problem was on their definitions. We will give our interpretation.

On based on the division by zero calculus, the meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 0/z$.

Of course, for any constant function $f(z) = c$, by the division by zero calculus,

$$\frac{f(z)}{z}|_{z=0} = 0.$$

For any c, we write it as follows:

$$\frac{z}{0} = 0,$$

as its definition. Here, we should not confuse with the result by the division by zero calculus

$$\frac{z}{z}|_{z=0} = 1.$$

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [88].

In order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero. It is a famous rule that we are not permitted to divide the numbers and functions by zero. For this old and general concept, we will give a simple and affirmative answer. In particular, certainly we gave several generalizations of division, however, we will wish to understand with some good feelings for **the division**. We wish to know with some good feelings for the sense of division. We wish to give a good meaning for the division by zero.

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{5.7}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \, |_{z=0} \, . \tag{5.8}$$

For example,

$$\frac{e^x}{x^n} \, |_{x=0} = \frac{1}{n!}.$$

In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

The content of this subsection was presented by [106].

5.10 Pythagorean theorem

In the well known parameter representation of the Pythagorean theorem for a right angle triangle

$$a = m + n,$$

$$b = \frac{2mn}{m - n},$$

and

$$c = \frac{m^2 + n^2}{m - n},$$

for the case of $m = n$, by the division by zero calculus, we have the interesting result

$$a = b = c = 2n.$$

5.11 Pompe's theorem

Theorem([95]): *Let ABC be an equilateral triangle and let G be a point on the side AB. Points P and Q lie on the sides AC and BC, respectively, and satisfy $\angle PGC = \angle QGC = \pi/6$. Let $\alpha = \angle AGP$ and $\beta = \angle BGQ$. Denote by r_1 and r_2 the inradii of the triangles AGP and BGQ, respectively. Then*

$$\frac{r_1}{r_2} = \frac{\sin 2\alpha}{\sin 2\beta}. \tag{5.9}$$

We consider the case $\beta = \pi/2$ in the sense of division by zero and division by zero calculus. In this case the point G coincides with B, then the triangle BQG degenerates to the point B, i.e., $r_2 = 0$. In this case the left side of (5.9) equals $r_1/0 = 0$. Also the right side equals $\sin 2\alpha/\sin 2\pi = \sin 2\alpha/0 = 0$. Therefore (5.9) holds.

On the other hand the right side of (5.9) is a function of B; $\sin 2(2\pi/3 - B)/\sin 2B$ and

$$\frac{\sin 2(2\pi/3 - x)}{\sin 2x} = -\frac{\sqrt{3}}{4x} + \frac{1}{2} + \frac{x}{\sqrt{3}} + \cdots .$$

This implies that

$$\frac{r_1}{r_2} = \frac{\sin 2\alpha}{\sin 2\beta} = \frac{1}{2}$$

in the case $\beta = 0$ by division by zero calculus. The large circle has radius $r_2 = 2r_1$ and center $B = Q$. It is orthogonal to the lines AB, BC and the perpendicular to AB at B. Therefore the circle still touches the three lines, since $\tan(\pi/2) = 0$, i.e., *it is the circle of radius $2r_1$ touching the lines AB, BC and the perpendicular to AB at B.*

Note that for many cases, we can calculate the division by zero calculus by MATHEMATICA, because it is just a coefficient of Laurent expansion.

For the beautiful figures and the details, see the original paper ([57]).

5.12 Remainder theorem and division by zero calculus

For the elementary theorem of remainder in polynomials we recall the division by zero calculus that appears naturally in order to show the importance of the division by zero calculus.

We found a very interesting question on the relation of the remainder theorem and division by zero on 2019.10.22 at Atsu Gake

Accademia Nuts: https://twitter.com/search?q=

On the good question, we stated our opinions to the site. We feel that the question is very natural and the problem may be contributed to a good understanding on the division by zero calculus.

The remainder theorem on polynomials may be stated as follows:

For a polynomial $f(x)$ when we look for the value $f(a)$, we divide it by the factor $(x-a)$ as follows:

$$\frac{f(x)}{x-a} = Q(x) \cdots R$$

with the remainder R. Then we obtain

$$f(a) = R.$$

Here, it seems that we divided there the function $f(x)$ by the zero $(x-a)|_{x=a}$ that was proposed as a question there. However, for the theorem there is no problem from the identity

$$f(x) = (x-a)Q(x) + R.$$

However, we would like to answer some general and good view-point on this problem.

Among any polynomials (or generally, analytic functions) we can consider the division; there we do not consider any singular points and zero points. Except singular points and zero points, of course, there is no problem for any division.

Now, by the division by zero calculus, we can consider the values at singular points and there is no problem in the logic for deriving the remainder theorem.

Indeed, in the identity

$$\frac{f(x)}{x-a}|_{x=a} = Q(x)|_{x=a} + \frac{R}{x-a}|_{x=a},$$

the identity

$$\frac{f(x)}{x-a}|_{x=a} = f'(a) = Q(x)|_{x=a}$$

is valid, by the division by zero calculus.

5.13 General order differentials and division by zero calculus

As a typical example, we recall the formula; for the function

$$y = \log x,$$

we have, for general order n derivatives,

$$y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}. \tag{5.10}$$

How will be the case for $n = 0$ in this formula? We will expect that for $n = 0$, $y = \log x$. However, in this case $(-1)!$ diverges as $\Gamma(0)$. Here, we will show that this curious property may be interpretated by the division by zero, precisely by the division by zero calculus.

By using the identity $(n-1)! = \Gamma(n)$ and we obtain, around $n = 0$, by considering an analytic function in n for $\Gamma(n)$

$$\Gamma(n) = \frac{1}{n} - \gamma + \frac{1}{12}(6\gamma^2 + \pi^2)n + ...$$

By the expansion

$$x^{-n} = \exp(-n \log x) = 1 - n \log x + \frac{1}{2} n^2 (\log x)^2 + \cdots,$$

we obtain the result, by the division by zero calculus

$$y^{(0)} = \log x + \gamma.$$

Here, the Euler constant γ appears in an extra way as in an integral constant.

For

$$y = \arctan x,$$

we have the formula

$$y^{(n)} = (n-1)! \cos^n y \sin n \left(y + \frac{\pi}{2} \right).$$

From the expansion

$$\cos^n y \sin n \left(y + \frac{\pi}{2} \right) = \left(y + \frac{\pi}{2} \right) n + () n^2 + \cdots,$$

we have

$$y + \frac{\pi}{2} = \arctan x + \frac{\pi}{2}.$$

We consider the function

$$y = a \arctan \frac{x}{a}.$$

Then, for $x > 0$

$$y^{(n)} = (-1)^{n-1} a \frac{(n-1)!}{(a^2 + x^2)^{(n/2)}} \sin \left(n \arctan \frac{a}{x} \right).$$

For $x < 0$

$$y^{(n)} = -a \frac{(n-1)!}{(a^2 + x^2)^{(n/2)}} \sin \left(n \arctan \frac{a}{x} \right).$$

From the expansion

$$a^{-n}\sin bn = (1 - n\log a + n^2() + \cdots)\left(bn - \frac{b^3n^3}{3!} + \cdots\right),$$

we have

$$y^{(n)} = -a\arctan\frac{a}{x} = a\left(\arctan\frac{x}{a} \pm \frac{\pi}{2}\right).$$

For the function

$$y = \arctan\frac{x\sin\alpha}{1 - x\cos\alpha},$$

we have

$$y^{(n)} = \frac{(n-1)!}{\sin^n\alpha}\sin n(\alpha + y)\sin^n(\alpha + y).$$

For the function

$$y = \arctan\frac{x\sin\alpha}{1 + x\cos\alpha},$$

we have

$$y^{(n)} = (-1)^{n-1}\frac{(n-1)!}{\sin^n\alpha}\sin n(\alpha - y)\sin^n(\alpha - y).$$

For these functions, from the expansion

$$a^{-n}\sin bn\sin^n(cn) = (1-n\log a+n^2()+\cdots)\left(bn - \frac{b^3n^3}{3!} + \cdots\right)$$

$$\cdot(1 - n\log c + n^2() + \cdots),$$

we obtain

$$y + \alpha,$$

and

$$y - \alpha,$$

respectively.

Meanwhile, for the function

$$f(x) = \arctan x,$$

we have

$$f'(x) = \frac{1}{x^2+1} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right).$$

Therefore, we have

$$f^{(n)}(x) = \frac{1}{2i}\left(\frac{\Gamma(n)}{(x-i)^n} - \frac{\Gamma(n)}{(x+i)^n}\right).$$

Then, by applying the division by zero calculus, we can consider negative integers n as integral operators.

In connection with the problem, we will give interesting examples.

For the function

$$y = \frac{ax+b}{cx+d},$$

we have, in general

$$y^{(n)} = (-1)^{n-1} n! \frac{(ad-bc)c^{n-1}}{(cx+d)^{n+1}}.$$

For $n = 0$, however, $y^{(0)} \neq y$.

For the function

$$y = x^3 \log \frac{x}{a},$$

we have, in general,

$$y^{(n)} = (-1)^{n-4} \frac{6(n-4)!}{x^{n-3}}. \tag{5.11}$$

For the case $n = 0$, by the expansion of $\Gamma(n-3)$ at $n = 0$

$$\Gamma(n-3) = -\frac{1}{6n} + \frac{1}{36}(6\gamma - 11) + ()n + \cdots,$$

we have

$$x^3 \log x + \frac{x^3}{60}(6\gamma - 11).$$

For the function

$$y_n = x^{n-1} \log x,$$

we have

$$y_n^{(n)} = \frac{(n-1)!}{x}.$$

Then, for $n = 0$, we have

$$-\frac{\gamma}{x}$$

and it is not y_0. However, they are valid for $n > 0$.

In general order n derivative representations of functions, when we consider negative orders, we have integral formulas for some case.

For example, in (5.10), when we use the expansions

$$\Gamma(n) = -\frac{1}{n+1} + (\gamma - 1) + ()(n+1) + \cdots$$

and

$$\frac{1}{x^n} = x - (n+1)x \log x + ()(n+1)^2 + \cdots,$$

we have the formula

$$\frac{1}{2}x^2 \log x + \frac{1}{4}(3 - 2\gamma)x^2.$$

In (5.11), from the expansions

$$\Gamma(n-3) = \frac{1}{24(n+1)} + \frac{1}{288}(25 - 12\gamma) + ()(n+1) + \cdots,$$

we obtain

$$\frac{1}{4}x^4 \log x - \frac{3}{144}(25 - 12\gamma)x^4.$$

For the function

$$y = \arctan \frac{1}{x},$$

we have

$$y^{(n)} = (-1)^n \Gamma(n) \sin ny \sin^n y.$$

For $n = 0$, by the division by zero calculus, we have $y^{(0)} = y$.

Meanwhile, for $n = -1$, by the division by zero calculus, we have

$$y^{(-1)} = \frac{y \cos y}{\sin y} - \log \sin y.$$

By noting that $y' = -\sin^2 y$, we see that

$$\left(y^{(-1)}\right)' = y.$$

Why division by zero for zero order representations for some general differential order representations of functions does happen?

5.14 Definition of division by zero calculus for multiply dimensions for differentiable functions

We would like to consider some general situation in Section 5.11, and we can consider in the following on some more abstract way on the line of Gâteaux differentiable and Frêchet differentiable functions. However, as the first step, we would like to consider the prototype case of the Taylor expansion in the three dimensional case as an essential and typical case.

We first recall the Taylor expansion

$$f(x, y, z)$$

$$= f(a, b, c) + \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right) f(a, b, c) +$$

$$+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right)^2 f(a, b, c) + \cdots$$

$$+\frac{1}{n!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}+(z-c)\frac{\partial}{\partial z}\right)^n f(a,b,c)+\cdots.$$

Then, in particular, note that as in the one dimensional way, for $R=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

$$f(x,y,z)=f(a,b,c)+$$

$$\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}+(z-c)\frac{\partial}{\partial z}\right)f(a,b,c)+o(R).$$

When we consider the case $R=0$, we should consider its direction as in

$$\lim_{R\to 0}\left(\frac{x-a}{R},\frac{y-b}{R},\frac{y-b}{R}\right)=(\ell,m,n).$$

We will denote the unit vector (ℓ,m,n) at (a,b,c) by $\mathbf{u}(a,b,c)$. Then, we **define** the division by zero calculus

$$\frac{f(x,y,z)}{R}|_{R=0}$$

by

$$(\nabla f)(a,b,c)\cdot\mathbf{u}(a,b,c);$$

that is,

$$\frac{f(x,y,z)}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}|_{(a,b,c)}=(\nabla f)(a,b,c)\cdot\mathbf{u}(a,b,c). \tag{5.12}$$

We can define a general order division by zero calculus

$$\frac{f(x,y,z)}{R^n}|_{R=0}$$

similarly.

As in one dimensional case, if a functions is not differentiable in the definition, then we shall define it as zero.

As in the one dimensional case, we can apply formulas for ∇ to the division by zero calculus. For example, in the formulas

$$\nabla(f+g) = \nabla f + \nabla g,$$

$$\nabla(fg) = g\nabla f + f\nabla g,$$

$$\nabla(f(g)) = \frac{df}{dg}\nabla g,$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B}),$$

and

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2},$$

for example, we obtain

$$\frac{f(x,y,z)g(x,y,z)}{R}|_{R=0}$$

$$= g(a,b,c)\frac{f(x,y,z)}{R}|_{R=0} + f(a,b,c)\frac{g(x,y,z)}{R}|_{R=0}.$$

Anyhow, with our definition, we can consider the division by zero calculus

$$\frac{f(x,y,z)}{R^n}|_{R=0}$$

that appears in many formulas.

Examples:

We shall examine examples.

1. The value of the function $y = x/|x|$ at $x = 0$ is ± 1 in our sense, here;

$$y = \frac{x}{|x|}|_{x=0} = \pm 1,$$

that depends on the unit vector $\mathbf{u}$.

However, since the division by zero calculus is not always almighty, we should consider the value 0 also;

$$y = \frac{x}{|x|}|_{x=0} = \frac{0}{0} = 0,$$

in some practical sense. Note that this function is an odd function and 0 is the mean value around the origin.

2.

$$\frac{x\exp(xy)}{\sqrt{(x-a)^2+(y-b)^2}}|_{(a,b,c)} = (\ell(1+ab)+ma^2)e^{ab}.$$

3.

$$\frac{x\exp(xy)}{\sqrt{(x-a)^2+(y-b)^2}}|_{(a,b,c)} = (\ell(1+ab)+ma^2)e^{ab}.$$

4.

$$\frac{\log\sqrt{x^2+y^2+z^2}}{R}|_{R=0} = \frac{\ell a+mb+nc}{a^2+b^2+c^2}.$$

5.

$$\frac{(x^2+y^2+z^2)^{n/2}}{R}|_{R=0}$$

$$= n(\ell a+mb+nc)\left(a^2+b^2+c^2\right)^{(n-2)/2}, n=-1,1,2,....$$

6.

$$\frac{f(\sqrt{x^2+y^2+z^2})}{R}|_{R=0} = \frac{\ell a+mb+nc}{\sqrt{a^2+b^2+c^2}}f'(\sqrt{a^2+b^2+c^2}).$$

7.

$$\frac{(x^2+y^2+z^2)\exp(-\sqrt{x^2+y^2+z^2})}{R}|_{R=0}$$

$$= (\ell a+mb+nc)\left(2-\sqrt{a^2+b^2+c^2}\right)\exp(-\sqrt{a^2+b^2+c^2}).$$

Green's functions:

For the fundamental solution of the Laplace equation

$$\Delta G(R) = -\delta(\mathbf{r}-\mathbf{r}'),$$

we have, for $R = |\mathbf{r} - \mathbf{r}'|$,

$$G_1(R) = -\frac{1}{2}R,$$

$$G_2(R) = -\frac{1}{2\pi}\log R$$

and

$$G_3(R) = -\frac{1}{4\pi R},$$

depending on the dimensions. We know that at the singular point $R = 0$, they are all zero.

For the fundamental solutions of the Helmholtz type equation

$$(\Delta + k^2)G(R) = -\delta(\mathbf{r} - \mathbf{r}'),$$

we have

$$G_1^{\pm}(R) = \frac{\pm i}{2k}\exp(\pm ikR)$$

and

$$G_3^{\pm}(R) = \frac{1}{4\pi R}\exp(\pm ikR).$$

depending on the dimensions and selections of branches. Then we have:

$$G_1^{\pm}(0) = \frac{\pm i}{2k}$$

and

$$G_3^{\pm}(R)|_{R=0} = \frac{\pm ik}{4\pi}.$$

For the fundamental solutions of the Klein-Gordon equation

$$(\Delta - \mu^2)G(R) = -\delta(\mathbf{r} - \mathbf{r}'),$$

we have

$$G_1(R) = \frac{1}{2\mu}\exp(-\mu R)$$

and

$$G_3(R) = \frac{1}{4\pi R}\exp(-\mu R),$$

depending on the dimensions. Then we have:

$$G_1(0) = \frac{1}{2\mu}$$

and

$$G_3(R)|_{R=0} = \frac{-\mu}{4\pi}.$$

In addition, we have:

$$\frac{\cos kR}{4\pi R}|_{R=0} = 0$$

and

$$\frac{\sin kR}{4\pi R}|_{R=0} = \frac{k}{4\pi}.$$

One dimensional case and multiply dimensional case:

We considered the division by zero calculus in the natural way for multiply dimensional cases. However, we see that its basic idea is similar with the one dimensional case, indeed, when we consider it on some line as in $x-a = \ell R, y-b = mR, z-c = nR$, we can see that the result is the same as the definition of the division by zero calculus in one dimensional case.

At this moment, we have another idea for (5.12); indeed, it looks like that the definition (5.12) is independent of the vector $\mathbf{u}(a,b,c)$. Following this idea, we can consider it as follows

$$\frac{f(x,y,z)}{R}|_{R=0} = (\nabla f)(a,b,c).$$

Whether this definition is good or bad will depend on the global properties of the division by zero calculus.

We shall consider one example in the multiply dimensional case.

On the complex plane, we shall consider the point η such that the two lines through (β and γ) and (α and η) are orthogonal and η is on the line through (β and γ).

Then, we obtain the formula

$$\eta = \frac{\overline{\alpha}(\beta - \gamma) + \alpha(\overline{\beta} - \overline{\gamma}) + \beta\overline{\gamma} - \overline{\beta}\gamma}{2(\beta - \gamma)}.$$

We will consider the case $\beta = \gamma$. Then, we obtain the formula, by the division by zero calculus for complex analysis

$$\eta = \frac{1}{2}(\overline{\alpha} + \overline{\gamma}).$$

Meanwhile, when we consider β on some line through the point γ, then we obtain the natural result that η is on the line and the line through (α and η) are orthogonal with the line.

We will consider the division by zero calculus for the several complex variable case in Section 13 that shows some more delicate properties.

5.15 Division by zero calculus in Ford circles

We will refer to an application of the division by zero calculus in Ford circles that have the relations to some criteria of irrational numbers as covering problems and to the Farey sequence F_n for any positive integer n.

Definitions of Ford circles

First we will recall Ford circles. Consider any two relatively prime integers h and k, then the circle $C(h,k)$ of radius $1/(2k^2)$ centered at $(h/k, 1/(2k^2))$ is known as a Ford circle. Let d be the distance between the centers of the circles with $C(h,k)$ and $C(h',k')$

$$d^2 = \left(\frac{h'}{k'} - \frac{h}{k}\right)^2 + \left(\frac{1}{2k'^2} - \frac{1}{2k^2}\right)^2$$

and s be the sum of the radii

$$s = r_1 + r_2 = \frac{1}{2k^2} + \frac{1}{2k'^2}.$$

Then

$$d^2 - s^2 = \frac{(h'k - hk')^2 - 1}{k^2 k'^2}.$$

From $d^2 - s^2 \geq 0$, $(h'k - k'h)^2 \geq 1$, and so the two circles are touching (tangency) if and only if

$$|h'k - k'h| = 1. \tag{5.13}$$

See ([37]).

Ford circles are related to the Farey sequence ([20], Conway and Guy 1996).

The Farey sequence F_n for any positive integer n is the set of irreducible rational numbers a/b with $0 \leq a \leq b \leq n$ and $(a, b) = 1$ arranged in increasing order. The first few are

$$F_1 = \{0/1, 1/1\}$$

$$F_2 = \{0/1, 1/2, 1/1\}$$

$$F_3 = \{0/1, 1/3, 1/2, 2/3, 1/1\}$$

$$F_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1\}$$

$$F_5 = \{0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1\}$$

and so on.

Except for F_1, each F_n has an odd number of terms and the middle term is always $1/2$.

Let p/q, p'/q', and p''/q'' be three successive terms in a Farey series. Then

$$qp' - pq' = 1$$

and

$$\frac{p'}{q'} = \frac{p + p''}{q + q''}. \tag{5.14}$$

This is the intermediate number of Farey.

If $h_1/k_1, h_2/k_2$, and h_3/k_3 are three consecutive terms in a Farey sequence, then the circles $C(h_1, k_1)$ and $C(h_2, k_2)$ are tangent at

$$\alpha_1 = \left(\frac{h_2}{k_2} - \frac{k_1}{k_2(k_2^2 + k_1^2)}, \frac{1}{k_1^2 + k_2^2} \right) \tag{5.15}$$

and the circles $C(h_2, k_2)$ and $C(h_3, k_3)$ intersect in

$$\alpha_2 = \left(\frac{h_2}{k_2} + \frac{k_3}{k_2(k_2^2 + k_3^2)}, \frac{1}{k_2^2 + k_3^2} \right).$$

Moreover, α_1 lies on the circumference of the semicircle with diameter $(h_1/k_1, 0) - (h_2/k_2, 0)$ and α_2 lies on the circumference of the semicircle with diameter $(h_2/k_2, 0) - (h_3/k_3, 0)$ ([4], Apostol 1997, p. 101).

The aim of this Section is to consider the special circle $C(h, 0)$ from the view point of the division by zero calculus. For this purpose, we will consider the group of the circles $C(h, k)$ for real numbers h, k (we do not consider the conditions of rational numbers and of co-primeness (h,k)=1).

The division by zero calculus is to consider the case $k = 0$ in the fractional h/k.

Then, how to consider h for $h/0 = 0$? On the above line and from the primeness $(h, k) = 1$, we would like to consider the case $h = 1$. Indeed, we would like to show that **the irruducible fraction of $h/0$ may be considered as $1/0$; that is $h = 1$.**

In this case, we will consider the property of Ford circles from the viewpoint of the division by zero calculus.

3 circles appear as the circle $C(1, 0)$

We will show that 3 circles appear as the circle $C(1, 0)$ from the division by zero calculus view point. We write $C(h, k)$ as follows:

$$\left(x - \frac{h}{k} \right)^2 + \left(y - \frac{1}{2k^2} \right)^2 = \left(\frac{1}{2k^2} \right)^2;$$

that is,

$$x^2 - 2\frac{h}{k}x + \left(\frac{h}{k}\right)^2 + y^2 - \frac{1}{k^2}y = 0. \tag{5.16}$$

Hence, by the division by zero calculus, we have, for $k = 0$, $x = y = 0$; this means that the circle $C(1,0)$ is the point circle and it is the origin

$$C(1,0) = \{0\}. \tag{5.17}$$

Next, from

$$x^2k - 2hx + \frac{h^2}{k} + y^2k - \frac{1}{k}y = 0, \tag{5.18}$$

we obtain, similarly

$$C(1,0) = \{x = 0\}. \tag{5.19}$$

Finally, from

$$x^2k^2 - 2hkx + h^2 + y^2k^2 - y = 0, \tag{5.20}$$

we obtain, similarly

$$C(1,0) = \{y = 1\}. \tag{5.21}$$

In the sequel, we will consider these three cases.

Case I

This point circle is a very natural case. In particular, note that a point circle may be considered as zero radius and zero curvature ([54]). Firstly, it may be considered as touching with the real line. Secondly, the condition (5.13) is valid for $k = 0, h = 1$ and note that in the case $k' = 0$; that means that there is no non-degenerate circles $C(h', k')$ touching with the origin point circle. The third condition (5.14) also is satisfied with the sense that the three circles all have to be the origin point circle.

α_1 property (5.15) is valid in the degenerated sense of $k_1, k_2 = 0$ and $\alpha_1 = 0$.

Case II

Firstly, note that $\tan(\pi/2) = 0$ and for some natural sense we can consider that the y axis and the x axis are orthogonal, however, they are, at the same time, touching each other; that is the gradients of the both lines are zero and the same. This property appeared in many cases, already. See, for example, ([54, 57, 84, 88]).

Any circle $C(h, k)$ touching with the x and y axes can be represented by the relation

$$h = \frac{1}{2k}.$$

Then, of course, we have

$$\frac{h}{k} = \frac{1}{2k^2}.$$

Therefore, with the parameter $k > 0$, when we consider two circles $C(1, 0)$ and $C(1/(2k), k)$, the property (5.13) is valid only with $k = 1$.

The reasons are on the facts that the center and radius of a line are the origin and zero, respectively, when we consider a line as a circle.

The property (5.14) is not valid.

α_1 property (5.15) is not valid.

Case III

In this case, we can consider that the both lines $y = 1$ and $y = 0$ are touching each other at the point at infinity. In this case, the situation is clear, because any circle $C(h, k)$ touching with the both lines is represented by

$$k^2 = 1.$$

Therefore, we see that in this case all the properties are valid.

α_1 property (5.15) is also valid.

In particular, note that, even this case, the center of the circle $C(1,0) = \{y = 1\}$ is the origin.

Remarks

The Ford circles have deep properties for some criteria of irrational numbers with covering problems as follows:

Theorem: *For a real number α, it is an irrational number if and only if there exist infinitely many numbers h/k; irruducible rational numbers satisfying the inequality*

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{2k^2}.$$

See, for example, ([20, 29, 41]).

As a general circle group of the Ford circles, we will consider

$$(x-\xi)^2 + (y - f(\xi))^2 = f(\xi)^2$$

with a differentiable function $f(\xi)$ around the origin. Then, by the same logic we obtain the three cases, similarly for $\xi = 0$

$$(I): \qquad x^2 + y^2 - 2f(0)y = 0,$$

$$(II): \qquad x + f'(0)y = 0$$

and

$$(III): \qquad 1 - f''(0)y = 0.$$

5.16 Division by zero calculus and computers

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here:
https://isabelle.in.tum.de/

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan \frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp \frac{1}{x} (x = 0) = 1,$$

and others.

The relation of Isabelle/HOL and division by zero is unclear at this moment, however, the following document will be interested in:

Dear Saitoh,

In Isabelle/HOL, we can define and redefine every function in different ways. So, logarithm of zero depends upon our definition. The best definition is the one which simplify the proofs the most. According to the experts, z/0 = 0 is the best definition for division by zero.

$$\tan(\pi/2) = 0$$

$$\log 0 =$$

is undefined (but we can redefine it as 0)

$$e^0 = 1$$

(but we can redefine it as 0)

$$0^0 = 1$$

(but we can redefine it as 0).

In the attached file you will find some versions of logarithms and exponentials satisfying different properties. This file can be opened with the software Isabelle/HOL from this webpage: https://isabelle.in.tum.de/

Kind Regards,

José M.

(2017.2.17.11:09).

At 2019.3.4.18:04 for my short question, we received:

It is as it was programmed by the HOL team.
Jose M.
On Mar 4, 2019, Saburou Saitoh wrote:
Dear José M.
I have the short question.
For your outputs for the division by zero calculus, for the input, is it some direct or do you need some program???
With best regards, Sincerely yours,
Saburou Saitoh 2019.3.4.18:00

Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [28],

he kindly sent the message:

> It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Surprisingly enough, he sent his e-mail at 2019.3.30.18:42 as follows:

> Nevertheless, you can use that $x/0 = 0$, following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL: https://isabelle.in.tum.de/
>
> and copying the following code
>
> theory DivByZeroSatoih imports Complex Main
>
> begin
>
> theorem T: ‹x/0 + 2000 = 2000› for x :: complex
> by simp

5.17 Division by zero calculus and Laplace transform

We will consider the Laplace transform from the viewpoint of the division by zero calculus with typical examples. The images of the Laplace transform are analytic functions on some half complex plane and meanwhile, the division by zero calculus gives some values at isolated singular points of analytic functions. Then, how will be the Laplace transform at the isolated singular points? For this basic question, we will be able to obtain a new concept for the Laplace integral. At the isolated singular points, of course, the Laplace transform (integral) does not exist in the usual sense and so the problem is delicate and new.

1. For the Laplace transform of the function
$$\frac{t^{n-1}e^{-at}}{(n-1)!}, \quad n = 1, 2, 3, ...,$$
we have
$$\frac{1}{(s+a)^n}.$$
Then, for $s = -a$, by the division by zero calculus (DBZC), we have
$$\frac{1}{(s+a)^n}(-a) = 0.$$
Then, how will be the corresponding Laplace transform
$$\int_0^\infty \frac{t^{n-1}e^{-at}}{(n-1)!} e^{at} dt = \int_0^\infty \frac{t^{n-1}}{(n-1)!} dt$$
? Note that this integral is zero, because infinity may be represented by 0. Conversely, from this argument for the general function for any positive k
$$\frac{\Gamma(k)}{(s+a)^k}$$
that is the Laplace transform of the function
$$t^{k-1}e^{-at},$$
we can derive the result
$$\frac{\Gamma(k)}{(s+a)^k}(-a) = 0.$$
Indeed, since this result is not defined by DBZC for general positive k, this result now was derived here, by this logic.

2. For the Laplace transform of the function

$$\frac{e^{-at} - e^{-bt}}{b-a}, \quad a < b$$

we have

$$\frac{1}{(s+a)(s+b)}.$$

Then, for $s = -a$, by DBZC, we have

$$\frac{1}{(s+a)(s+b)}(-a) = -\frac{1}{(b-a)^2}.$$

Then, the corresponding Laplace transform

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{b-a} e^{at} dt = \frac{1}{b-a} \int_0^\infty \left(1 - e^{-(b-a)t}\right) dt$$
$$= -\frac{1}{(b-a)^2},$$

that is right.

3. For the Laplace transform of the function

$$\frac{ae^{-at} - be^{-bt}}{a-b}, \quad a < b$$

we have

$$\frac{s}{(s+a)(s+b)}.$$

Then, for $s = -a$, by DBZC, we have

$$\frac{s}{(s+a)(s+b)}(-a) = \frac{b}{(b-a)^2}.$$

Then, the corresponding Laplace transform

$$\int_0^\infty \frac{ae^{-at} - be^{-bt}}{a-b} e^{at} dt = \frac{1}{a-b} \int_0^\infty \left(a - be^{-(b-a)t}\right) dt$$
$$= \frac{b}{(b-a)^2},$$

that is right.

4. For the Laplace transform of the function

$$\frac{1}{a}\sinh at$$

we have

$$\frac{1}{(s-a)(s+a)}.$$

Then, for $s = a$, by DBZC, we have

$$\frac{1}{(s-a)(s+a)}(a) = -\frac{1}{4a^2}.$$

Then, the corresponding Laplace transform

$$\frac{1}{2a}\int_0^\infty \left(1 - e^{-2at}\right) dt = -\frac{1}{4a^2},$$

that is right.

5. For the Laplace transform of the function

$$\cosh at$$

we have

$$\frac{s}{(s-a)(s+a)}.$$

Then, for $s = a$, by DBZC, we have

$$\frac{s}{(s-a)(s+a)}(a) = \frac{1}{4a}.$$

Then, the corresponding Laplace transform

$$\frac{1}{2}\int_0^\infty \left(1 + e^{-2at}\right) dt = \frac{1}{4a},$$

that is right.

6. For the Laplace transform of the function

$$\frac{1}{a^3}\left(at - \sin at\right)$$

we have

$$\frac{1}{s^2(s^2+a^2)}.$$

Then, for $s = 0$, by DBZC, we have

$$\frac{1}{s^2(s^2+a^2)}(0) = -\frac{1}{a^4}.$$

Then, the corresponding Laplace transform

$$\frac{1}{a^3}\int_0^\infty \left(at - \sin at\right) e^{-0t} dt = -\frac{1}{a^4},$$

that is right. However, here note that

$$\int_0^\infty \sin at dt = \frac{1}{a},$$

in the sense of distribution theory.

7. For the Laplace transform of the function

$$\frac{1}{a^2}(1 - \cos at),$$

we have

$$\frac{1}{s(s^2+a^2)}.$$

Then, for $s = 0$, by DBZC, we have

$$\frac{1}{s(s^2+a^2)}(0) = 0.$$

Then, the corresponding Laplace transform

$$\frac{1}{a^2}\int_0^\infty (1 - \cos at)e^{-0t} dt = 0,$$

that is right.

8. For the step function $u(t)$, the Laplace transform of the function $u(t-k)$ is given by

$$\frac{1}{s}e^{-ks}.$$

Then, by DBZC, we have

$$\left(\frac{1}{s}e^{-ks}\right)(0) = -k.$$

Then, its Laplace transform is

$$\int_k^\infty e^{0t}dt = [t]_k^\infty = -k,$$

that is right. Note that $\infty = 0$.

9. The Laplace transform of the function $(t-k)u(t-k)$ is given by

$$\frac{1}{s^2}e^{-ks}.$$

Then, by DBZC, we have

$$\left(\frac{1}{s^2}e^{-ks}\right)(0) = \frac{k^2}{2}.$$

Then, its Laplace transform is

$$\int_k^\infty (t-k) = \left[\frac{t^2}{2} - kt\right]_k^\infty = \frac{k^2}{2},$$

that is right.

10. For the Laplace transform of the function

$$1 - 3e^{-t} + 3e^{-2t}$$

we have

$$\frac{s^2+2}{s(s+1)(s+2)}.$$

Then, for $s = 0$, by DBZC, we have

$$\frac{s^2 + 2}{s(s+1)(s+2)}(0) = -\frac{3}{2}.$$

Note that by the theory of Oliver Heaviside we can calculate the inverse Laplace transform of the form

$$\frac{p(s)}{q(s)}$$

that is for polynomials $p(s), q(s)$.

Then, the corresponding Laplace transform

$$\int_0^\infty \left(1 - 3e^{-t} + 3e^{-2t}\right) dt = -\frac{3}{2},$$

that is right.

11. For the Laplace transform of the function

$$-\gamma - \log t,$$

we have

$$\frac{1}{s} \log s.$$

Then, for $s = 0$, we have

$$\left(\frac{1}{s} \log s\right)(0) = 0.$$

Note that by the general definition of the division by zero calculus for differentiable functions, this result is derived also in this way

$$\left(\frac{1}{s} \log s\right)(0) = (\log s)'(0) = \left(\frac{1}{s}\right)(0) = 0.$$

In general, we obtain

$$\left(\frac{1}{s^k} \log s\right)(0) = 0, \quad k > 0.$$

Of course, we can derive many and many examples.

For the Dirac delta distribution δ, we have

$$\delta(\omega) = \frac{1}{\pi}\int_0^\infty \cos \omega t dt.$$

Then we see that

$$\delta(0) = 0.$$

By taking derivative, we have

$$\delta'(\omega) = \frac{1}{\pi}\int_0^\infty -t \sin \omega t dt.$$

Hence,

$$\delta'(0) = 0.$$

In general, we obtain that

$$\delta^{(n)}(0) = 0, \quad n = 0, 1, 2, 3, ...$$

We are interested in some definite statement for the relation of the division by zero calculus and Laplace integrals.

5.18 Division by zero calculus and spectral theory for closed operators

We will discuss the spectral theory for closed operators from the viewpoint of the division by zero calculus. If μ is isolated in the spectrum σ of a closed operator $A : D(A) \subset X \to X$, the resolvent $R(\lambda, A)$ of A at the point λ can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{\infty} (\lambda - \mu)^n U_n$$

on a ring domain $0 < |\lambda - \mu| < \delta$ with some small $\delta > 0$. The coefficients U_n of this expansion are bounded operators. In particular, U_{-1} is the spectral projection P corresponding to the decomposition $\sigma(A) = \{\mu\} \cup (\sigma(A) \setminus \mu)$ of the spectrum of A.

In general, we have, for any $n, m > 0$

$$U_{(n+1)} = (A - \mu)^n P$$

and

$$U_{(n+1)} \cdot U_{(m+1)} = U_{(n+m+1)}.$$

See [33], Chapter 4 for the details.

Then, by the division by zero calculus, we can consider the operator U_0 that is corresponding to the Laurent coefficient for $n = 0$ as a specially important case as in U_{-1}.

5.19 From electromagnetism

Ichiroh Fujimoto obtained the following very interesting definition (interpretation) at 2021.2.28.16:40 from the idea of Wolhard Hövel at 2021.2.15.15:04 based on a simple physical model on electromagnetism.

For any positive integer n and for a positive real number R, we define

$$f_R(z) = \begin{cases} \frac{1}{z^n} & \text{for} \quad |z| > R \\ \frac{\overline{z}^n}{R^{2n}} & \text{for} \quad |z| \leq R. \end{cases}$$

Note that the above two functions have the analytic extensions over the circle with the radius R and with its center the origin.

Then, we see that

$$\lim_{R \to 0} f_R(z) = \begin{cases} \frac{1}{z^n} & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0. \end{cases}$$

5.20 Remarks for the applications of the division by zero and the division by zero calculus

As the number system, we can calculus the arithmetic by the Yamada field structure. However, for functions, the problems are involved for their structure. So, for applying the division by zero calculus, we should consider and apply the division by zero and division by zero calculus in many ways and check the results. **By considering many ways, we will be able to see many new aspects and results obtained. By checking the results obtained, we will be able to find new prospects. With this idea, we can enjoy the division by zero calculus with a free spirit without logical problems.** – For this idea, we may ask and consider what is mathematics?

6 TRIANGLES, TRIGONOMETRIC FUNCTIONS AND HARMONIC MEAN

In order to see how elementary of the division by zero, we will see the division by zero in triangles, trigonometric functions and harmonic mean as the fundamental objects. Even the cases of triangles, trigonometric functions and harmonic mean, we can derive new concepts and results.

Even the case

$$\tan x = \frac{\sin x}{\cos x},$$

we have the identity, for $x = \pi/2$

$$0 = \frac{1}{0}.$$

Of course, for identities for analytic functions, they are still valid even at isolated singular points with the division by zero calculus. Here, we will see many more direct applications of the division by zero $1/0 = 0/0 = 0$ as in the above with some meanings.

Note that from the inversion of the both sides

$$\cot x = \frac{\cos x}{\sin x},$$

for example, we have, for $x = 0$,

$$0 = \frac{1}{0}.$$

By this general method, we can consider many problems.

For the identity

$$\tan\left(\theta + \frac{\pi}{2}\right) = \frac{-1}{\tan\theta},$$

for the case $\theta = 0$, we have

$$\tan\frac{\pi}{2} = \frac{-1}{\tan 0} = 0,$$

(Hisao Inoue in Quora, 2020.6.11).

We will consider a triangle ABC with $BC = a, CA = b, AB = c$. Let θ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and we did not consider that by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

In the Napier's formula

$$\frac{a+b}{a-b} = \frac{\tan(A+B)/2}{\tan(A-B)/2},$$

there is no problem for $a = b$ and $A = B$.

Masakazu Nihei derived the result (H. Okumura sent his result at 2018.11.29.10:06):

Let θ be the angle of ADB for the midpoint D of BC. Then, we have

$$\tan\theta = \frac{2bc\sin A}{(b-c)(b+c)}.$$

Here, for $b=c$, of course, we have $\theta = \pi/2$ and $\tan\frac{\pi}{2} = 0$.

Similarly, in the formula

$$\frac{b-c}{b+a}\frac{1}{\tan\frac{A}{2}} + \frac{b+c}{b-c}\tan\frac{A}{2} = \frac{2}{\sin(B-C)},$$

for $b=c$, $B=C$, and we have

$$0 + \frac{2c}{0}\tan\frac{A}{2} = \frac{2}{0},$$

that is right.

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2+b^2-c^2=0$, then by the Pythagorean theorem $C=\pi/2$. Then,

$$0 = \frac{\tan B}{\tan\frac{\pi}{2}} = \frac{\tan B}{0}.$$

Meanwhile, for the case $a^2-b^2+c^2=0$, $B=\pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0} = \frac{\tan\frac{\pi}{2}}{\tan C} = 0.$$

In the formula

$$\frac{a^2+b^2+c^2}{2abc} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c},$$

for the case $a=0$, with $b=c$ and $B=C=\pi/2$ the identity holds.

Meanwhile, the lengths f and f' of the bisector lines of A and in the out of the triangle ABC are given by

$$f = \frac{2bc\cos\frac{A}{2}}{b+c}$$

and

$$f' = \frac{2bc\sin\frac{A}{2}}{b-c},$$

respectively.

If $b = c$, then we have $f' = 0$, by the division by zero. However, note that, from

$$f' = 2\sin\frac{A}{2}\left(c + \frac{c^2}{b-c}\right),$$

by the division by zero calculus, for $b = c$, we have

$$f' = 2b\sin\frac{A}{2} = a.$$

The result $f' = 0$ is a popular property, but the result $f' = a$ is also an interesting popular property. See [57].

We will consider the two lines, for $0 < b < a$

$$y = \tan\frac{\theta_a}{2}(x-a)$$

and

$$y = \tan\frac{\theta_b}{2}(x-b).$$

Then, the common point is given by

$$\left(\frac{a\tan\frac{\theta_a}{2} - b\tan\frac{\theta_b}{2}}{\tan\frac{\theta_a}{2} - \tan\frac{\theta_b}{2}}, (a-b)\frac{\sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}}{\sin\frac{\theta_a-\theta_b}{2}}\right).$$

Note that this is the center of the inscribed circle of the triangle ABC; $A = (a,0), B = (b,0)$ with respect to A that is in the out side of the line AC and its radius of the circle is given by the y of the center.

For $\theta_a = \theta_b$, by the division by zero calculus, we have that the common point is the origin

$$(0,0)$$

and $y = 0$ (H. Okumura: 2020.3.14.10:07).

We will consider a triangle $\triangle ABC; \angle CBA = \theta_b, BA = c, AC = b, CB = a, \angle CAB = \pi - \theta_a, B(q, 0), A(p, 0)$. Then, the point C is represented by

$$\left(\frac{p \tan \theta_a - q \tan \theta_b}{\tan \theta_a - \tan \theta_b}, \frac{c \sin \theta_a \sin \theta_b}{\sin(\theta_a - \theta_b)} \right),$$

$$a = \frac{c \sin \theta_b}{\sin(\theta_a - \theta_b)}$$

and

$$b = \frac{c \sin \theta_a}{\sin(\theta_b - \theta_a)}.$$

For $\theta_a = \theta_b$, we have

$$C = (0, 0), a = b = 0$$

(H. Okumura: 2020.3.14.10:07).

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively. Let θ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan \theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

In the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

if b or c is zero, then, by the division by zero, we have $\cos A = 0$. Therefore, then we should understand as $A = \pi/2$.

This result may be derived from the formulas

$$\sin^2 \frac{A}{2} = \frac{(a - b + c)(a + b - c)}{4bc}$$

and

$$\cos^2 \frac{A}{2} = \frac{(a + b + c)(-a + b + c)}{4bc},$$

by applying the division by zero calculus.

This result is also valid in the Mollweide's equation

$$\sin \frac{B-C}{2} = \frac{(b-c)\cos \frac{A}{2}}{a},$$

for $a = 0$ as

$$0 = \frac{(b-c)\cos \frac{A}{2}}{0}.$$

Let r be the radius of the inscribed circle of the triangle ABC, and r_A, r_B, r_C be the distances from A, B, C to the lines BC, CA, AB, respectively. Then we have

$$\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

When A is the point at infinity, then, $r_A = 0$ and $r_B = r_C = 2r$ and the identity still holds.

We have the identities, for the radius R of the circumscribed circle of the triangle ABC,

$$S = \frac{ar_A}{2} = \frac{1}{2}bc\sin A$$

$$= \frac{1}{2}a^2\frac{\sin B \sin C}{\sin A}$$

$$= \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = rs, \quad s = \frac{1}{2}(a+b+c).$$

If A is the point at infinity, then, $S = s = r_A = b = c = 0$ and the above identities all valid.

For the identity

$$\tan \frac{A}{2} = \frac{r}{s-a},$$

if the point A is the point at infinity, $A = 0, s - a = 0$ and the identity holds as $0 = r/0$. Meanwhile, if $A = \pi$, then the identity holds as $\tan(\pi/2) = 0 = 0/s$.

In the identities

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2}$$

and

$$\cot A + \cot B + \cot C = \cot A \cdot \cot B \cdot \cot C + \csc A \cdot \csc B \cdot \csc C,$$

we see that they are valid for $A = \pi$ and $B = C = 0$.

For the identity

$$\cot(z_1 + z_2) = \frac{\cot z_1 \cot z_2 - 1}{\cot z_1 + \cot z_2},$$

for $z_1 = z_2 = \pi/4$, the identity holds.

For a triangle, we have the identity

$$\cot A + \cot(B + C) = 0.$$

For the case $A = \pi/2$, the identity is valid.

For the identity

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C,$$

for $A = \pi/2$, the identity is valid.

In the sine theorem:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

for $A = \pi$, $B = C = 0$ and then we have

$$\frac{a}{0} = \frac{b}{0} = \frac{c}{0} = 0.$$

In the formula

$$\frac{\cos A \cos B}{ab} + \frac{\cos B \cos C}{bc} + \frac{\cos C \cos A}{ca} = \frac{\sin^2 A}{a^2}$$

for $a = 0$, $A = 0$, $b = c$, $B = C = \pi/2$, the identity is valid.

In the formula

$$R = \frac{abc}{4S},$$

for $S = 0$, we have

$$R = 0$$

(H. Okumura: 2017.9.5.7:40).

In the formula

$$\cos A + \cos B = \frac{2(a+b)}{c} \sin^2 \frac{C}{2},$$

for $c = 0$, we have $b = c$ and $A = B = \pi/2$ and the identity is valid.

In a triangle ABC, let H be the orthocenter and J be the common point of the three perpendicular bisectors. Then, we have

$$AH = d_a = \frac{a}{\tan A}$$

and

$$\text{the distance of } J \text{ to the line } BC = h_a = \frac{a}{2\tan A}.$$

For $A = \pi/2$, we have that $d_a = h_2 = 0$ (V. V. Puha: 2018.7.12.18:10).

In a triangle ABC, let X be the leg of the perpendicular line from A to the line BC and let Y be the common point of the bisector line of A and the line BC. Let P and Q be the tangential points on the line BC with the incircle of the triangle and the inscribed circle in the sector with the angle A, respectively. Then, we know that

$$\frac{XP}{PY} = \frac{XQ}{QY}.$$

If $AB = AC$, then, of course, X=Y=P=Q. Then, we have

$$\frac{0}{0} = \frac{0}{0} = 0.$$

Let X, Y, Q be the common points with a line and three lines AC, BC and AB, respectively. Let P be the common point with the line AB and the line through the point C and the common point of the lines AY and BX. Then, we know the identity

$$\frac{AP}{AQ} = \frac{BP}{BQ}.$$

If two lines XY and AB are parallel, then the point Q may be considered as the point at infinity. Then, by the interpretation $AQ = BQ = 0$, the identity is valid as

$$\frac{AP}{0} = \frac{BP}{0} = 0.$$

For the tangential function, note the following identities.

In the formula

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}},$$

for $\theta = \pi$, we have that 0=0/0.

For the inversions and from $x = 0$, we have

$$\frac{1}{0} = \frac{2}{0} = \pm\sqrt{\frac{2}{0}} = 0.$$

In the formula

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 + z_2)}{\sin z_1 \sin z_2},$$

for $z_1 = \pi/2, z_2 = 0$, we have that 0=1/0.

In the formula

$$\tan\frac{x}{2} = \frac{1 \pm \sqrt{1-\sin^2 x}}{\sin x},$$

for $x = \pi$, we have

$$0 = \frac{0}{0}.$$

In the elementary identity

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta},$$

for the case $\alpha = \beta = \pi/4$, we have

$$\tan\frac{\pi}{2} = \frac{1+1}{1-1\cdot 1} = \frac{2}{0} = 0.$$

Further note that it is valid for $\alpha+\beta=0$ and $\alpha+\beta=\pi/2$ (H. Okumura: 2018.8.7.8:03). Furthermore, the formula

$$\tan(\alpha+\beta)=\frac{\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}}{\frac{1}{\tan\alpha\tan\beta}-1},$$

is valid for $\alpha=\pi/2$ or $\beta=\pi/2$ and for $\alpha+\beta=\pi/2$ (H. Okumura: 2018.7.11.21:05).

In the identity

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}=\frac{1}{\cos\alpha}-\tan\alpha,$$

for $\alpha=\pi/2$, we have

$$0=\frac{1}{0}-0.$$

For the double angle formula

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha},$$

for $\alpha=\pi/2$, we have that

$$0=\frac{2\cdot 0}{1-0}.$$

In the identity

$$\tan 3\alpha=\frac{2\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha},$$

for $\alpha=\pi/6$, we have

$$\tan\frac{\pi}{2}=0,$$

that is right.

In the identities

$$\frac{1+\cos x}{\sin x}=\frac{\sin x}{1-\cos x}$$

and

$$\frac{1-\cos x}{\sin x}=\tan\frac{x}{2}$$

for $x=0$, we have the identity

$$\frac{0}{0}=0.$$

In the identities

$$\tan(x+iy)=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

and

$$\cot(x+iy)=\frac{\sin 2x-i\sinh 2y}{\cosh 2x-\cos 2x}$$

for $x=\pi/2, y=0$, they are valid.

In the identity

$$\arctan\sqrt{\frac{b}{a}}=\arcsin\sqrt{\frac{b}{a+b}},$$

for $a=0$, the identity is valid.

We can find similar interesting identities in the following identities:

$$\frac{\sin 3x+\sin x}{\cos 3x+\cos x}=\tan 2x,$$
$$\frac{\sin 3x+\sin x}{\cos 3x-\cos x}=-\cot x$$

and

$$\frac{\sin 3x-\sin x}{\cos 3x+\cos x}=\tan x$$

(H. Okumura: 2019.1.31 and the first one is obtained by M. Nihei).

We consider a triangle ABP with $A(-a,0)$, $B(a,0)$, $P(x,y)$ with $a>0$. Then we have

$$\angle QPA=\tan^{-1}\frac{x+a}{y}$$

and

$$\angle QPB = \tan^{-1}\frac{y-a}{y}.$$

For $y=0$, we obtain

$$\angle QPA = \tan^{-1}\frac{x+a}{0} = \tan^{-1}0 = \frac{\pi}{2}$$

and

$$\angle QPB = \tan^{-1}\frac{x-a}{0} = \tan^{-1}0 = \frac{\pi}{2}.$$

In addition, M. Nihei remarked the following identities through H. Okumura (2018.7.8.12:12; 2019.1.31):

$$\frac{\sin 2x}{1+\cos x} = \frac{1-\cos x}{\sin 2x} = \tan x,$$

$$\frac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x} = \cot x,$$

$$\frac{\sin 2x}{1-\cos 2x} = \frac{1+\cos 2x}{\sin 2x} = \cot x,$$

and in a triangle

$$\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4S}.$$

Consider the case $x=\pi/2$ in the above two formulas and in the triangle, consider the case $A=\pi/2$.

We recall that the harmonic mean $H(a,b)$ for non zero real numbers a,b is given by

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}, \tag{6.1}$$

or

$$H(a,b) = \frac{2ab}{a+b}. \tag{6.2}$$

Here, we wish to consider, for example, for $b=0$. When we use the representation (6.2), then we have $H(a,0)=0$, however, when we use the presentation (6.1), we have $H(a,0) = 2a$, by the division by zero. We would like to show that the result in this case should be that

$$H(a,0) = a. \tag{6.3}$$

Our interpretation for this result is given by the following way.

At first, by the division by zero

$$H(a,0) = \frac{2}{\frac{1}{a} + \frac{1}{0}} = 2a.$$

However, in this case the number 2 means the harmonic mean for two numbers and in this case 0 means that the zero term does not exist; that is, nothing or void and so, we should replace by one for two. Then, we will have our interpretation. In the sequel, we will show this interesting phenomena in several geometric properties which show interesting new phenomena on Euclidean geometry.

Quadrilateral case

The common point of the line $y = bx$ and $y = -ax + a$ $(a, b > 0)$ is $(a/(a+b), ab/(a+b))$.

For the function $y = bx$, if $b = 0$, then we have $y = 0$ and we have $H/2 = 0$.

However, for the function $y = bx$, if $b = 0$, from $y/b = x$ and the division by zero, we have $x = 0$ and then, $H/2 = a$.

Triangle case

We consider the line passing two points $(a, 0)$ and $(0, b)$ $(a, b > 0)$. Then for the distance h from the origin to the line is given by

$$h^2 = \frac{a^2b^2}{a^2 + b^2}.$$

For the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

for $b = 0$, we have $x = a$, by the division by zero. Then, we can consider that $h = a$. Meanwhile, from the equation $bx + ay = ab$ and for $b = 0$, we have $y = 0$. Then, we have $h = 0$.

Trapezoid case

We consider the trapezoid surrounded by the 4 lines y axis, x axis, $y = 2r$ $(r > 0)$ and the tangential line of the circle $(x - r)^2 + (y - r)^2 = r^2$. Let $(a, 0)$ be the common point with the x axis and the tangential line and let $(b, 2r)$ be the common point with the tangential line and the line $y = 2r$. Then, we have

$$H = 2r = \frac{2ab}{a + b}.$$

The inscribed circle in the trapezoid is given by

$$(a + b)^2(x^2 + y^2) - 2ab(a + b)(x + y) + a^2b^2 = 0.$$

By dividing by b^2, we have, by the division by zero

$$(x - a)^2 + (y - a)^2 = a^2. \tag{6.4}$$

By dividing by b, we have, by the division by zero

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2. \tag{6.5}$$

For $b = 0$, we have

$$x^2 + y^2 = 0. \tag{6.6}$$

In the cases (6.4) and (6.6), $H/2$ may be looked as a and 0, respectively and for (6.5), the result may be looked curiously.

Semi-circle case

For a fixed $a > 0$ and for $b > 0$, we consider the semi-circle

$$x^2 - (a + b)x + y^2 = 0. \tag{6.7}$$

The distance to the line connecting the points $((a+b)/2, 0)$ and the common point with the circle and the line $x = a$ from the point $(a, 0)$ is given by

$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}. \tag{6.8}$$

For $b = 0$, we have the circle

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2. \tag{6.9}$$

This means that $H = 0$.

However, by dividing the circle by b, by the division by zero, we have $x = 0$; this means that $H/2 = a$.

For the harmonic mean, its source is taken from [87].

V. V. Puha gave the beautiful definition of the harmonic means with the good notation as follows:

$$H(x_1, x_2, ..., x_n) = \frac{\sum_{j=1}^{n} \frac{x_j}{x_j}}{\sum_{j=1}^{n} \frac{1}{x_j}}$$

(2018.6.4.7:22).

For the formula, for any positive constants

$$\frac{ab(a' + b')}{aa' + bb' + cc'};$$

which has a beautiful geometry interpretation, for $a = a', b' = 0$ it is b. However, from

$$\frac{ab(1/b' + 1/a')}{a/b' + b/a' + 1};$$

or

$$\frac{ab(a'/b' + 1)}{aa'/b' + b + a'};$$

by the division by zero, we have

$$\frac{ab}{a+b}$$

which has a beautiful relation with the general case (H. Okumura: 2019.3.9.8:40).

7 DERIVATIVES OF A FUNCTION

On differential coefficients (derivatives), we obtain new concepts, from the division by zero calculus. At first, we will consider the fundamental properties. From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{7.1}$$

or

$$f'(x) = -\infty, \tag{7.2}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{7.3}$$

This definition is reasonable, because the point at infinity is represented by 0.

This property was also derived from the fact that the gradient of the y axis is zero; that is,

$$\tan \frac{\pi}{2} = 0, \tag{7.4}$$

that was looked from many geometric properties in Section 6 and in [57], and also in the formal way from the result $1/0 = 0$. Of course, by the division by zero calculus, we can derive analytically the result, because

$$\tan x = -\frac{1}{x - \pi/2} + \frac{1}{3}(x - \pi/2) + \frac{1}{45}(x - \pi/2)^3 + \cdots .$$

From the reflection formula of the Psi (Digamma) function

$$\psi(1 - z) = \psi(z) + \pi \frac{1}{\tan \pi z}$$

([1], 258), we have, for $z = 1/2$,

$$\tan \frac{\pi}{2} = 0.$$

Note also from the identity

$$\frac{1}{\psi(1-z)-\psi(z)} = \frac{\tan \pi z}{\pi},$$

we have

$$\frac{1}{\psi(1-z)-\psi(z)}(z=0) = 0$$

and

$$\frac{1}{\psi(1-z)-\psi(z)}\left(z=\frac{\pi}{2}\right) = 0.$$

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1-x^2},$$

$$y' = \frac{-x}{\sqrt{1-x^2}},$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0.$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta, y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\left(\frac{1}{z-\nu\pi}+\frac{1}{\nu\pi}\right) \tag{7.5}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

Note that in (7.5), since

$$\left(\frac{1}{z-\nu\pi}+\frac{1}{\nu\pi}\right)_{\nu=0}=\frac{1}{z},$$

we can write it simply

$$\cot z = \sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-\nu\pi}+\frac{1}{\nu\pi}\right).$$

We note that in many and many formulas we can apply this convention and modification.

The differential equation

$$y' = -\frac{x}{y}$$

with a general solution

$$x^2 + y^2 = a^2$$

is satisfied for all points of the solutions by the division by zero. However, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1$$

are not satisfied for the points $(-a, 0)$ and $(a, 0)$.

In many and many textbooks, we find the differential equations, however, they are not good in this viewpoint.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{7.6}$$

and so,

$$[y']_{x=0} = 0. \tag{7.7}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{7.8}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant C

$$y = \log x + C, \tag{7.9}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{7.10}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all values are zero at the origin, in the sense of the division by zero calculus.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable in the usual sense, but it has a singularity at the origin. For $x > 0$, there is no problem for (7.8) and (7.9). At $x = 0$, we see that we can not consider the limit in the usual sense. However, for $x > 0$ we have (7.9) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{7.11}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have the identity

$$\left[(\log x)'\right]_{x=0} = 0$$

and we will be able to understand its sense graphically.

Note that the function

$$y = ax + b + \frac{1}{x}$$

and its derivative

$$y' = a - \frac{1}{x^2}.$$

Then, the tangential approximate line at $x = 0$ of the function is the y axis and so the gradient of the function at the origin may be considered as zero, however, the derivative at the origin in the our sense at the singular point is a.

However, note that the gradients of the tangential lines of the curve converge to a when x tends to $+\infty$, and the origin and the point at infinity are coincident; that is the curve has two tangential lines at the origin (at the point at infinity) and their gradients are zero and a.

By the new interpretation for the derivative, we can arrange the formulas for derivatives, by the division by zero. The formula

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} \tag{7.12}$$

is very fundamental. Here, we considered it for a local one to one correspondence of the function $y = f(x)$ and for nonvanishing of the denominator

$$\frac{dy}{dx} \neq 0. \tag{7.13}$$

However, if a local one to one correspondence of the function $y = f(x)$ is ensured like the function $y = x^3$ around the origin, we do not need the assumption (7.13). Then, for the point $dy/dx = 0$, we have, by the division by zero,

$$\frac{dx}{dy} = 0.$$

This will mean that the function $x = g(y)$ has the zero derivative and the tangential line at the point is a parallel line to the y-axis. In this sense the formula (7.12) is valid, even the case $dy/dx = 0$. The nonvanising case, of course, the identity

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \tag{7.14}$$

holds. When we put the vanishing case, here, we obtain the identity

$$0 \times 0 = 1, \tag{7.15}$$

in a sense. Of course, it is not valid, because (7.14) is unclear for the vanishing case. Such an interesting property was referred to by M. Yamane in ([51]).

In addition, for higher-order derivatives, we note the following properties. For a function $y = f(x) \in C^3$ whose higher-order derived functions of the inverse function $x = g(x)$ are single-valued, we note that the formulas

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}$$

and

$$\frac{d^3x}{dy^3} = -\left[\frac{d^3y}{dx^3}\frac{dy}{dx} - 3\left(\frac{d^2y}{dx^2}\right)^2\right]\left(\frac{dy}{dx}\right)^{-5}$$

are valid, even at a point x_0 such that

$$f(x_0) = y_0, f'(x_0) = 0$$

as

$$\frac{d^2x}{dy^2}(y_0) = \frac{d^3x}{dy^3}(y_0) = 0.$$

Furthermore, the formulas

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2},$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3},$$

$$\left(\frac{1}{f}\right)''' = \frac{6ff'f'' - 6(f')^3 - f^2f'''}{f^4},$$

..., and so on, are valid, even the case

$$f(x_0) = 0,$$

at the point x_0.

In those identities in the framework of analytic functions, at first we consider their formulas except singular points and then, following the definition of division by zero calculus at singular points we consider the valid identities. For the case of functions that are not analytic functions, we have to consider case by case at singular points by division by zero or division by zero calculus idea and we have to check the results.

The derivative of the function

$$f(x) = \sqrt{x}(\sqrt{x} + 1)$$

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}}(\sqrt{x} + 1) + \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}} \end{aligned}$$

is valid at even the origin by using the function $\frac{\sqrt{x}}{\sqrt{x}}$ (V. V. Puha: 2018. June). He derived such formulas by using the function x/x.

In particular, note that the division by zero calculus is not almighty. The notation

$$\Delta(x) = \frac{x}{x} = x \cdot \frac{1}{x} = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \neq 0 \end{cases}$$

will be convenient in connection with the Dirac delta function $\delta(x)$.

Thales' theorem

We consider a triangle BAC with $A(-1,0), C(1,0), \angle BOC = \theta; O(0,0)$ on the unit circle. Then, the gradients of the lines AB and CB are given by

$$\frac{\sin\theta}{\cos\theta + 1}$$

and

$$\frac{\sin\theta}{\cos\theta - 1},$$

respectively. We see that for $\theta = \pi$ and $\theta = 0$, they are zero, respectively.

Indeed,

$$\frac{\sin x}{\cos x + 1} = -\frac{2}{x - \pi} + \frac{x - \pi}{6} + \cdots,$$

and

$$\frac{\sin x}{\cos x - 1} = -\frac{2}{x} + \frac{x}{6} + \cdots.$$

Implicit functions

In the function $y = y(x)$ defined by a differentiable implicit function $f(x, y) = 0$, we have the formula

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

If $f_y(a, b) = 0$, then the tangential line through the point (a, b) of the function is given by

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0,$$

that is

$$x = a.$$

Then we have

$$\frac{dy}{dx}(a, b) = -\frac{f_x(a, b)}{0} = 0.$$

Differential quotients and division by zero

We will refer to an interesting interpretation of the relation between differential and division by zero.

For the differential quotient

$$\frac{dy}{dx},$$

if it is zero in some interval, then, of course, we have that $y = C$ in the interval with a constant C. This will mean that if $dy = 0$, then $y = C$ in some interval with a constant C and $y' = 0$.

Meanwhile, if $dx = 0$, then, by the division by zero, we have

$$\frac{dy}{dx} = 0$$

and so, we have that $y' = 0$. Then, however, $x = D$ with a constant D in some y interval. This interpretation shows that the gradient of the y axis is zero, that is

$$\tan\frac{\pi}{2} = 0.$$

8 DIFFERENTIAL EQUATIONS

From the viewpoint of the division by zero calculus, we will see many incomplete results in mathematics, in particular, in the theory of differential equations at an undergraduate level; indeed, we have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we considered our mathematics with the limiting concept, however, the limiting value to the singular point and the value at the singular point of the function are, in general, different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. From this viewpoint, we will be able to consider differential equations even at singular points. We find many incomplete statements and problems in many undergraduate textbooks. In this section, we will point out the problems in concrete ways by examples.

This section is an arrangement of the papers [3] and [96] with new materials.

8.1 Missing a solution

For the differential equation

$$2xydx - (x^2 - y^2)dy = 0,$$

we have a general solution with a constant C

$$x^2 + y^2 = 2Cy.$$

However, we are missing the solution $y = 0$. By this expression

$$\frac{x^2 + y^2}{C} = 2y,$$

for $C = 0$, by the division by zero, we have the missing solution $y = 0$.

For the differential equation

$$x(y')^2 - 2yy' - x = 0,$$

we have the general solution

$$C^2x^2 - 2Cy - 1 = 0.$$

However, $x = 0$ is also a solution, because

$$xdy^2 - 2ydydx - xdx^2 = 0.$$

From

$$x^2 - \frac{2y}{C} - \frac{1}{C^2} = 0,$$

by the division by zero, we obtain the solution.

For the differential equation

$$2y = xy' - \frac{x}{y'},$$

we have the general solution

$$2y = Cx^2 - \frac{1}{C}.$$

For $C = 0$, we have the solution $y = 0$, by the division by zero.

For the differential equation

$$(x^2 - a^2)(y')^2 - 2xyy' - x^2 = 0,$$

we have the general solution

$$y = Cx^2 - \left(a^2C + \frac{1}{4C}\right).$$

For $C = 0$, $y = 0$, however, this is not a solution. But, this is the solution of the differential equation

$$(x^2 - a^2)\frac{(y')^2}{y} - 2xy' - \frac{x^2}{y} = 0.$$

For the differential equation

$$ydx + (x^2y^3 + x)dy = 0,$$

we have the general solution

$$-\frac{1}{xy} + \frac{y^2}{2} = C.$$

Of course, we have the solution $y = 0$.

For the differential equation

$$(3x^2 - 1)dy - 3xydx = 0,$$

we have the general solution

$$3x^2 + 2 = Cy^2 + 3.$$

From

$$\frac{3x^2 + 2}{C} = y^2 + \frac{3}{C},$$

we have the solution $y = 0$, by the division by zero.

8.2 Differential equations with singularities

For the differential equation

$$y' = -\frac{y}{x},$$

we have the general solution

$$y = \frac{C}{x}.$$

From the expression

$$xdy + ydx = 0,$$

we have also the general solution

$$x = \frac{C}{y}.$$

Therefore, there is no problem for the origin. Of course, $x = 0$ and $y = 0$ are the solutions.

For the differential equation

$$y' = \frac{2x - y}{x - y}, \tag{8.1}$$

we have the beautiful general solution with constant C

$$2x^2 - 2xy + y^2 = C. \tag{8.2}$$

By the division by zero calculus we see that on the whole points on the solutions (8.2) the differential equation (8.1) is satisfied. If we do not consider the division by zero, for $y = x(\neq 0)$, we will have a serious problem. However, for $x = y \neq 0$, we should consider that $y' = 0$, not by the division by zero calculus, but by $1/0 = 0$.

For the differential equation

$$y' = \frac{2xy}{x^2 - y^2}$$

and for the general solution

$$x^2 + (y - C)^2 = C^2,$$

there is no problem at the singular points $y = x$.

For the differential equation

$$xy' = y^2 + y,$$

we have the general solution with constant C

$$y' = -\frac{x}{x - C}.$$

At the point $x = C$, the equation is satisfied by the division by zero $1/0 = 0$, not the division by zero calculus.

For the differential equation

$$x^3 y' = x^4 - x^2 y + 2y^2,$$

we have the general solution with constant C

$$y = \frac{x^2(x+C)}{2x+C}. \tag{8.3}$$

Note that we have also a solution $x = 0$, because,

$$x^3 dy = (x^4 - x^2 y + 2y^2)dx.$$

In particular, note that at $(0,0)$

$$y'(0) = \frac{0}{0},$$

and the general solution (8.3) has the value

$$y\left(-\frac{1}{2}C\right) = -\frac{1}{8}C^2,$$

by the division by zero calculus. For C tending to ∞ in the general solution, we have the solution $y = x^2$. Then, if we understand $C = 0$, we see that the property of the solution is valid.

8.3 Continuation of solution

We will consider the differential equation

$$\frac{dx}{dt} = x^2 \cos t. \tag{8.4}$$

Then, as the general solution, we obtain, for a constant C

$$x = \frac{1}{C - \sin t}.$$

For $x_0 \neq 0$, for any given initial value (t_0, x_0) we obtain the solution satisfying the initial condition

$$x = \frac{1}{\sin t_0 + \frac{1}{x_0} - \sin t}. \tag{8.5}$$

If

$$\left|\sin t_0 + \frac{1}{x_0}\right| < 1,$$

then the solution has many poles and L. S. Pontrjagin stated in his book that the solution is disconnected at the poles and so, the solution may be considered as infinitely many solutions.

However, from the viewpoint of the division by zero, the solution takes the value zero at the singular points and the derivatives at the singular points are all zero; that is, the solution (8.5) may be understood as one solution.

Furthermore, by the division by zero, the solution (8.5) has its sense for even the case $x_0 = 0$ and it is the solution of (8.4) satisfying the initial condition $(t_0, 0)$.

We will consider the differential equation

$$y' = y^2.$$

For $a > 0$, the solution satisfying $y(0) = a$ is given by

$$y = \frac{1}{\frac{1}{a} - x}.$$

Note that the solution satisfies on the whole space $(-\infty, +\infty)$ even at the singular point $x = \frac{1}{a}$, in the sense of the division by zero, as

$$y'\left(\frac{1}{a}\right) = y\left(\frac{1}{a}\right) = 0.$$

8.4 Singular solutions

We will consider the differential equation

$$(1 - y^2)dx = y(1 - x)dy.$$

By the standard method, we obtain the general solution, for a constant C $(C \neq 0)$

$$\frac{(x-1)^2}{C} + y^2 = 1.$$

By the division by zero, for $C = 0$, we obtain the singular solution

$$y = \pm 1.$$

For the simple Clairaut differential equation

$$y = px + \frac{1}{p}, \quad p = \frac{dy}{dx},$$

we have the general solution

$$y = Cx + \frac{1}{C}, \tag{8.6}$$

with a general constant C and the singular solution

$$y^2 = 4x.$$

Note that we have also the solution $y = 0$ from the general solution, by the division by zero $1/0 = 0$ from $C = 0$ in (8.6).

8.5 Solutions with singularities

1). We will consider the differential equation

$$y' = \frac{y^2}{2x^2}.$$

We will consider the solution with an isolated singularity at a point a taking the value $-2a$ in the sense of division by zero.

First, by the standard method, we have the general solution, with a constant C

$$y = \frac{2x}{1 + 2Cx}.$$

From the singularity, we have, $C = -1/2a$ and we obtain the desired solution

$$y = \frac{2ax}{a - x}.$$

Indeed, from the expansion

$$\frac{2ax}{a - x} = -2a - \frac{2a^2}{x - a},$$

we see that it takes $-2a$ at the point a in the sense of the division by zero calculus. This function was appeared in ([56]).

2). For any fixed $y > 0$, we will consider the differential equation

$$E(x,y)\frac{\partial E(x,y)}{\partial x} = \frac{y^2 d^2}{(y-x)^3}$$

for $0 \leq x \leq y$. Then, note that the function

$$E(x,y) = \frac{y}{y-x}\sqrt{d^2 + (y-x)^2}$$

satisfies the differential equation satisfying the condition

$$[E(x,y)]_{x=y} = 0,$$

in the sense of the division by zero. This function was appeared in showing a strong discontinuity of the curvature center (the inversion of EM diameter) of the circle movement of the rotation of two circles with their radii x and y in ([56]).

3). We will consider the singular differential equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} - \frac{3}{x^2}y = 0. \tag{8.7}$$

By the series expansion, we obtain the general solution, for any constants a, b

$$y = \frac{a}{x^3} + bx. \tag{8.8}$$

We see that by the division by zero

$$y(0) = 0, y'(0) = b, y''(0) = 0.$$

The solution (8.8) has its sense and the equation is satisfied even at the origin. The value $y'(0) = b$ may be given arbitrary, however, in order to determine the value a, we have to give some value for the regular point $x \neq 0$. Of course, we can give the information at the singular point with the Laurent coefficient

a, that may be interpreted with the value at the singular point zero. Indeed, the value a may be considered at the value

$$[y(x)x^3]_{x=0} = a.$$

4). Next, we will consider the Euler differential equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0.$$

We obtain the general solution, for any constants a, b

$$y = \frac{a}{x} + \frac{b}{x^2}.$$

This solution is satisfied even at the origin, by the division by zero and furthermore, all derivatives of the solution of any order are zero at the origin.

5). We will note that as the general solution with constants C_{-2}, C_{-1}, C_0

$$y = \frac{C_{-2}}{x^2} + \frac{C_{-1}}{x} + C_0,$$

we obtain the ordinary differential equation

$$x^2y''' + 6xy'' + 6y' = 0.$$

6). For the differential equation

$$y' = y^2(2x - 3),$$

we have the special solution

$$y = \frac{1}{(x-1)(2-x)}$$

on the interval $(1, 2)$ with the singularities at $x = 1$ and $x = 2$. Since the general solution is given by, for a constant C,

$$y = \frac{1}{-x^2 + 3x + C},$$

we can consider some conditions that determine the special solution.

8.6 Solutions with an analytic parameter

For example, in the ordinary differential equation

$$y'' + 4y' + 3y = 5e^{-3x},$$

in order to look for a special solution, by setting $y = Ae^{kx}$ we have, from

$$y'' + 4y' + 3y = 5e^{kx},$$

$$y = \frac{5e^{kx}}{k^2 + 4k + 3}.$$

For $k = -3$, by the division by zero calculus, we obtain

$$y = e^{-3x}\left(-\frac{5}{2}x - \frac{5}{4}\right),$$

and so, we can obtain the special solution

$$y = -\frac{5}{2}xe^{-3x}.$$

For example, for the differential equation

$$y'' + a^2 y = b\cos\lambda x,$$

we have a special solution

$$y = \frac{b}{a^2 - \lambda^2}\cos\lambda x.$$

Then, for $\lambda = a$ (reasonance case), by the division by zero calculus, we obtain the special solution

$$y = \frac{bx\sin(ax)}{2a} + \frac{b\cos(ax)}{4a^2}.$$

Indeed, we have the expansion

$$y = -\frac{b\cos(ax)}{2a(\lambda - a)} + \left(\frac{bx\sin(ax)}{2a} + \frac{b\cos(ax)}{4a^2}\right) + (\cdot)(\lambda - a) + \cdots.$$

The Newton kernel, for $N > 2$,

$$\Gamma_N(x,y) = \frac{1}{N(2-N)\omega_N}|x-y|^{2-N}$$

and

$$\Gamma_2(x,y) = \frac{1}{2\pi}\log|x-y|,$$

where

$$\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}.$$

From $\Gamma_N(x,y)$, by the division by zero calculus, we have

$$\frac{1}{2\pi}\log|x-y| + \frac{1}{4\pi}(\gamma + \log\pi),$$

where γ is the Euler constant.

For the Green function $G_N(x,y)$ of the Laplace operator on the ball with its center at a and its radius r on the Euclidean space of $N (N \geq 3)$ dimension is given by

$$G_N(x,y) = \|x-y\|^{2-N} - \left(\frac{r}{\|y-a\|}\frac{1}{\|x-y^*\|}\right)^{N-2},$$

where y^* is the inversion of y

$$y^* - a = \left(\frac{r}{\|y-a\|}\right)^2 (y-a).$$

By $N = 2$, we obtain the corresponding formula, by the division by zero calculus,

$$G_2(x,y) = \log\left(\frac{\|y-a\|}{r}\frac{\|x-y^*\|}{\|x-y\|}\right)$$

([5], page 91).

We can find many examples.

8.7 Special reductions by division by zero of solutions

For the differential equation

$$y'' - (a+b)y' + aby = e^{cx}, c \neq a, b; a \neq b,$$

we have the special solution

$$y = \frac{e^{cx}}{(c-a)(c-b)}.$$

If $c = a(\neq b)$, then, by the division by zero calculus, we have

$$y = \frac{xe^{ax}}{a-b}.$$

If $c = a = b$, then, by the division by zero calculus, we have

$$y = \frac{x^2 e^{ax}}{2}.$$

For the differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0,$$

we obtain the general solution, for $\gamma^2 > 4mk$

$$x(t) = e^{-\alpha t}\left(C_1 e^{\beta t} + C_2 e^{-\beta t}\right)$$

with

$$\alpha = \frac{\gamma}{2m}$$

and

$$\beta = \frac{1}{2m}\sqrt{\gamma^2 - 4mk}.$$

For $m = 0$, by the division by zero calculus we obtain the reasonable solution $\alpha = 0$ and $\beta = -k/\gamma$.

We will consider the differential equation, for a constant K

$$y' = Ky.$$

Then, we have the general solution

$$y(x) = y(0)e^{Kt}.$$

For the differential equation

$$y' = Ky\left(1 - \frac{y}{R}\right),$$

we have the solution

$$y = \frac{y(0)e^{Kt}}{1 + \frac{y(0)(e^{Kt}-1)}{R}}.$$

If $R = 0$, then, by the division by zero, we obtain the previous result, immediately.

We will consider the fundamental ordinary differential equation

$$x''(t) = g - kx'(t)$$

with the initial conditions

$$x(0) = -h, x'(0) = 0. \tag{8.9}$$

Then we have the solution

$$x(t) = \frac{g}{k}t + \frac{g(e^{-kt} - 1)}{k^2} - h.$$

Then, for $k = 0$, we obtain, immediately, by the division by zero calculus

$$x(t) = \frac{1}{2}gt^2 - h.$$

For the differential equation

$$x''(t) = g - k(x'(t))^2$$

satisfying the same condition with (8.9), we obtain the solution

$$x(t) = \frac{1}{2k} \log \frac{\left(e^{2t\sqrt{kg}} + 1\right)^2}{4e^{2t\sqrt{kg}}} - h.$$

Then, for $k = 0$, we obtain

$$x(t) = \frac{1}{2} g t^2 - h.$$

immediately, by the division by zero calculus.

For the differential equation

$$m x''(t) = -mg - r x'(t),$$

the solution satisfying the conditions $x(0) = x_0, x'(0) = v_0$ is given by

$$x(t) = -\frac{g}{r} m t + A + B \exp\left(-\frac{r}{m} t\right),$$

with

$$A = x_0 - B, B = -\frac{m}{r}\left(\frac{m}{r} g + v_0\right).$$

For $r = 0$, by the division by zero calculus, we have the reasonable solution

$$x(t) = -\frac{1}{2} g t + v_0 t + x_0.$$

For the differential equation

$$x''(t) = -g + k(x'(t))^2$$

satisfying the initial conditions

$$x(0) = 0, x'(0) = V,$$

we have

$$x'(t) = -\sqrt{\frac{g}{k}} \tan(\sqrt{kg} t - \alpha),$$

with

$$\alpha = \tan^{-1} \sqrt{\frac{k}{g}} V$$

and the solution

$$x(t) = \frac{1}{k} \log \frac{\cos\left(\sqrt{kg}t - \alpha\right)}{\cos \alpha}.$$

Then we obtain for $k = 0$, by the division by zero calculus

$$x'(t) = -gt + V$$

and

$$x(t) = -\frac{1}{2}gt^2 + Vt.$$

We will consider the typical ordinary differential equation

$$mx''(t) = mg - m(\lambda x'(t) + \mu(x'(t))^2),$$

satisfying the initial conditions

$$x(0) = x'(0) = 0.$$

Then we have the solution

$$x(t) = \frac{-\lambda + \sqrt{\lambda^2 + 4\mu g}}{2\mu}t+$$

$$\frac{1}{\mu}\log[(\frac{-\lambda + \sqrt{\lambda^2 + 4\mu g}}{2\mu}\exp(-\sqrt{\lambda^2 + 4\mu g t})$$

$$+\frac{\lambda + \sqrt{\lambda^2 + 4\mu g}}{2\mu})\frac{\mu}{\sqrt{\lambda^2 + 4\mu g}}].$$

Then, if $\mu = 0$, we obtain, immediately, by the division by zero calculus

$$x(t) = \frac{g}{\lambda}t + \frac{1}{\lambda^2}ge^{-\lambda t} - \frac{g}{\lambda^2}.$$

Furthermore, if $\lambda = 0$, then we have

$$x(t) = \frac{1}{2}gt^2.$$

We can find many and many such examples. However, note the following fact.

For the differential equation

$$y''' + a^2 y' = 0,$$

we obtain the general solution, for $a \neq 0$

$$y = A \sin ax + B \cos ax + C.$$

For $a = 0$, from this general solution, how can we obtain the corresponding solution

$$y = Ax^2 + Bx + C,$$

naturally?

For the differential equation

$$y' = ae^{\lambda x} y^2 + afe^{\lambda x} y + \lambda f,$$

we obtain a special solution, for $a \neq 0$

$$y = -\frac{\lambda}{a} e^{-\lambda x}.$$

For $a = 0$, from this solution, how can we obtain the corresponding solution

$$y = \lambda f x + C,$$

naturally?

8.8 Uchida's hyper exponential functions

K. Uchida ([126]) has a long love for the solutions of the differential equations

$$\frac{d^n y}{dx^n} = f(x) y$$

and he are appointing the importance of the solutions. He called the solutions *hyper exponential functions* (*Uchida's hyper exponential functions*). He considered the solutions for some functions $f(x)$ and derived many beautiful computer graphics with their elementary properties ([127]). We see the few concrete solutions from [94] and [127]. Of course, the case $n = 1$ is trivial and the $n > 3$ cases are rare examples and the case $n = 2$ is important.

Meanwhile, we can consider analytic functions and their derivatives even at isolated singular points. Therefore, we can consider the Uchida's exponential functions for analytic functions $f(x)$ with singularities. Surprisingly enough, then any analytic functions with any singular points may be considered as the Uchida's hyper exponential functions. As one typical example, we will consider the simplest case of

$$f(x) = \frac{1}{(x-a)^m} \tag{8.10}$$

for the general real number m of $m \neq 0$ and for $n = 2$.

We recall the general result that for the second order differential equation of the homogeneous equation

$$y'' + f_1(x)y' + f_0(x)y = 0 \tag{8.11}$$

and for a non-trivial solution $y_1 = y_1(x)$, the general solution is given by

$$y = y_1 \left(C_1 + C_2 \int \frac{\exp(-F(x))}{y_1(x)^2} dx \right); \quad F(x) = \int f_1(x) dx \tag{8.12}$$

(see, for example, [94], page 21).

Now, for any analytic function $f(x)$ with arbitrary singularities, of course, it satisfies the normal equation

$$\frac{d^2y}{dx^2} = \frac{f''(x)}{f(x)} y. \tag{8.13}$$

Furthermore, it is a nontrivial and simple solution for nonconstant function case of the homogeneous equation

$$y'' + \frac{f''(x)}{f'(x)} y' - \frac{2f''(x)}{f(x)} y = 0. \tag{8.14}$$

This type equation has the simple and non-trivial solution $f(x)$, and its structure may be checked with for the coefficient $f_1(x)$ of y'

$$f_1(x) = \frac{f''(x)}{f'(x)};$$

that is,

$$f'(x) = \exp\left(\int f_1(x)dx\right)$$

and

$$-\frac{2f''(x)}{f(x)} = \frac{-2f_1(x)\exp\left(\int f_1(x)dx\right)}{\int \left(\exp\int f_1(x)dx\right)dx}$$

should be the coefficient f_2 of y.

For example, we have the differential equations

$$y'' - (\tan x)y' + 2y = 0,$$

$$y'' - \left(2x + \frac{1}{x}\right)y' - 4(2x^2 + 1)y = 0,$$

$$y'' + \frac{x^2 - 4x + 1}{x(x^2 + 1)}y' + \frac{4(x^2 - 4x + 1)}{(x^2 + 1)^2}y = 0,$$

$$y'' - \frac{1}{x}y' + \frac{2}{x^2 \log x}y = 0,$$

and

$$y'' + \left(\frac{-1}{x} + \frac{1}{x\log x}\right)y' + \left(\frac{4}{x^2\log x} - \frac{1}{x^2(\log x)^2}\right)y = 0.$$

For $y_1(x) = f(x)$, when there exist the integrals, for

$$F(x) = \int f_1(x)dx = \log f'(x)$$

$$y_2(x) = y_1\int \frac{\exp(-F)}{y_1^2}dx = f(x)\int \frac{1}{f(x)^2 f'(x)}dx, \qquad (8.15)$$

the function y_2 is an independing solution of the equation (8.14).

Then, we know that the general solution is given by

$$y = C_1 y_1 + C_2 y_2 \tag{8.16}$$

$$+y_2 \int y_1 g \frac{dx}{W} - y_1 \int y_2 g \frac{dx}{W}.$$

Here W is the Wronskian determinant

$$W(x) = y_1 y_2' - y_2 y_1'$$

and it is, in general, given by Liouville's formula

$$W(x) = W(x_0) \exp\left[-\int_{x_0}^{x} f_1(t)dt\right] = \frac{1}{f'(x)}.$$

See [94], pages 21-23.

Then, for any continuous function g that is integrable in the following integrals, we obtain the general solution of the inhomogeneous differential equation

$$y'' + \frac{f''(x)}{f'(x)} y' - \frac{2f''(x)}{f(x)} y = g(x), \tag{8.17}$$

$$y = C_1 y_1 + C_2 y_2 \tag{8.18}$$

$$+y_2 \int f(x) f'(x) g(x) dx - y_1 \int y_2(x) f'(x) g(x) dx.$$

In connection with (8.14) and (8.15), we have the homogeneous equation, for $\alpha + \beta = -1$

$$y'' + \frac{\alpha f''(x)}{f'(x)} y' + \frac{\beta f''(x)}{f(x)} y = 0. \tag{8.19}$$

Therefore, for this case, we obtain the similar results.

For the simple case of $y = (x-a)^{\nu}$

For the function

$$y = (x-a)^{\nu}, \tag{8.20}$$

we obtain the normal equation

$$y'' = \frac{\nu(\nu-1)}{(x-a)^2}y. \tag{8.21}$$

Then, we see that the function

$$y = (x-a)^{\nu-1/2} \tag{8.22}$$

is a special solution of the differential equation

$$y'' + \frac{1}{x-a}y' - \frac{\nu(\nu-1)+(1/4)}{(x-a)^2}y = 0. \tag{8.23}$$

Therefore, by the above general formula, we obtain the general solution of (8.23) for $\nu \neq 1/2$,

$$y = C_1(x-a)^{\nu-(1/2)} + C_2(x-a)^{-\nu+(1/2)}. \tag{8.24}$$

For the case $\nu = 1/2$, the result and the situation are trivial.

For $\nu \neq 1/2$, since we obtain a fundamental system of the solutions

$$y_1(x) = (x-a)^{\nu-1/2}, \quad y_2(x) = (x-a)^{-\nu+(1/2)}, \tag{8.25}$$

of the homogeneous equation (8.23), we can obtain easily the general solution of the inhomogeneous equation

$$y'' + f_1(x)y' + f_0(x)y = g(x). \tag{8.26}$$

Here we assume that $a < x_0 < x_2, x_1 < x_0$. In our situation, we obtain the result:

For the inhomogeneous equation

$$y'' + \frac{1}{x-a}y' - \frac{\nu(\nu-1)+(1/4)}{(x-a)^2}y = g(x), \tag{8.27}$$

we obtain the general solution, for $\nu \neq 1/2$

$$y = C_1y_1 + C_2y_2 \tag{8.28}$$

$$+\frac{1}{-2\nu+1}\left[y_2\int(x-a)^{\nu+(1/2)}g(x)dx - y_1\int(x-a)^{(3/2)-\nu}g(x)dx\right],$$

if the integrals exist.

Thomas-Fermi equation and open problems

In connection with the normal equations, we recall the important Thomas-Fermi equation

$$y'' = \frac{1}{\sqrt{x}}y^{(3/2)}$$

in the Thomas-Fermi model. We find one special solution

$$y = \frac{144}{x^3}.$$

However, in connection with the theory of Uchida and with our new idea in this paper, the problems are all open.

See [94], pages 306-314, for many concrete examples of

$$y'' = Ax^n y^m$$

and their special solutions.

In [1], page 362, we see the following interesting equations and concrete special solutions:

$$y'' = -\left(\lambda^2 - \frac{\nu^2-(1/4)}{z^2}\right)y,$$

$$y'' = -\left(\frac{\lambda^2}{4z} - \frac{\nu^2-1}{4z^2}\right)y,$$

$$y'' = -\lambda^2 z^{p-1} y,$$

$$y'' = -(\lambda^2\exp(2z) - \nu^2)y,$$

and

$$y^{2n} = (-1)^n\lambda^{2n}z^{-n}y.$$

Remark

For the hyper exponential functions of the second order, note that:

For any C^2 function $f(x)$ on a closed interval that is a non-vanishing $f(x) \neq 0$, it is a solution of the normal equation (3.3) and conversely, for any C^2 function $h(x)$ on the closed interval, the normal differential equation

$$\frac{d^2y}{dx^2} = h(x)y$$

has a non-vanishing solution $f(x)$ of C^2 functions.

8.9 Partial differential equations

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bx\frac{\partial w}{\partial x} + (cx+d)w,$$

we have a special solution

$$w(x,t) = \exp\left[-\frac{c}{b}x + \left(d + \frac{ac^2}{b^2}\right)t\right].$$

For $b = 0$, how will be the corresponding solution? If $b = 0$, then $c = 0$ and

$$\frac{c}{b} = \frac{0}{0} = 0,$$

and we obtain the corresponding solution.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (be^{\beta t} + c)w,$$

we have special solutions

$$w(x,t) = (Ax+B)\exp\left[\frac{b}{\beta}e^{\beta t} + ct\right],$$

$$w(x,t) = A(x^2 + 2at)\exp\left[\frac{b}{\beta}e^{\beta t} + ct\right],$$

and

$$w(x,t) = A\exp\left[\lambda x + a\lambda^2 t + \frac{b}{\beta}e^{\beta t} + ct\right].$$

Then, we see that for $\beta = 0$, by the interpretation

$$\left[\frac{1}{\beta}e^{\beta t}\right]_{\beta=0} = t,$$

we can obtain the corresponding solution.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bxe^{\beta x} + c)w,$$

we have a special solution

$$w(x,t) = A\exp\left[\frac{b}{\beta}xe^{\beta t} + \frac{ab^2}{2\beta^3}e^{2\beta t} + ct\right].$$

Then, for $\beta = 0$, by the interpretation

$$\left[\frac{1}{\beta^j}e^{\beta t}\right]_{\beta=0} = \frac{1}{j!}t^j,$$

we can obtain the corresponding solution.

However, the above properties will be, in general, complicated.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw,$$

we have the fundamental solution

$$w(x,t) = \frac{1}{2\sqrt{\pi a t}}\exp\left(-\frac{x^2}{4at} + bt\right).$$

For $a = 0$, we have the corresponding solution

$$w(x,t) = \exp bt.$$

For the factor

$$\frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at}\right)$$

we have, for letting $a \to 0$,

$$\delta(x),$$

meanwhile, at $a = 0$, by the division by zero calculus, we have 0 that is a solution of the corresponding equation for $a = 0$. So, the reduction problem is a delicate open problem.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (-bx^2 + ct + d)w,$$

we have a special solution

$$w(x,t) = \exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}x^2 + \frac{1}{2}ct^2 + (\sqrt{ab} + d)t\right].$$

For $a = 0$, how will be the corresponding solution? Since we have the solution

$$w(x,t) = \exp\left[-bx^2 t + \frac{1}{2}ct^2 + dt\right],$$

for the factor

$$\frac{1}{2}\sqrt{\frac{b}{a}}x^2$$

we have to have

$$-bx^2 t.$$

8.10 Hadamard's example - ill-posed problems

The following example is famous with its ill-posed problem in general inverse problems:

We will consider the initial value problem of the Laplace operator $\Delta u = 0$ satisfying the initial conditions

$$u(x, 0) = 0, \quad u_y(x, 0) = g_n(x) = \frac{\sin nx}{n^2}.$$

Then, we obtain the solution

$$u(x, y) = \frac{\sinh y \sin nx}{n^2}, \quad n = 1, 2, 3, ...$$

Of course, for $n = 0$, we have the trivial solution, by the division by zero calculus, because $g_0(x) = 0$ and $u(x, y) = 0$.

8.11 Open problems

As important open problems, we would like to propose them clearly. We have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and the values at the singular point in the sense of division by zero calculus are, in general, different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. We thus have a general open problem discussing our mathematics on a domain containing the singular point.

We referred to the reduction problem by concrete examples; there we found the delicate property. For this interesting property we expect some general theory.

9 EUCLIDEAN SPACES AND DIVISION BY ZERO

In this section, we will see the division by zero properties on the Euclidean spaces. Since the impact of the division by zero and division by zero calculus is widely expanded in elementary mathematics, here, elementary topics will be introduced as the first stage.

9.1 Broken phenomena of figures by area and volume

The strong discontinuity of the division by zero around the point at infinity will appear as the destruction of various figures. These phenomena may be looked in many situations as the universal one. However, the simplest cases are the disc and sphere (ball) with their radius $1/\kappa$. When $\kappa \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $\kappa = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry, as we see later. However, the results are clear already from the definition of the division by zero.

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$ (see [2]). We thus see that

$$\lim_{z \to \infty} z = \infty, \tag{9.1}$$

however,

$$[z]_{z=\infty} = 0, \tag{9.2}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (9.1) and (9.2) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x \to +\infty} x = +\infty, \qquad \lim_{x \to -\infty} x = -\infty,$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0.$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [57], we gave many examples for this property.

In particular, in $z \to \infty$ in (9.1), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (9.1) represents the limit by means of the ϵ - δ logic. That is, for any large number M, when we take for some large number N, we have, for $|z| > N$, $|z| > M$.

9.2 Parallel lines

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2.$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right).$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the common point is always given by

$$(0, 0),$$

even two lines are the same. This fact shows that the image of the Euclidean space in Section 3 is right.

In particular, note that the concept of parallel lines is very important in the Euclidean plane and non-Euclidean geometry.

With our sense, there are no parallel lines and all lines pass the origin. This will be our world on the Euclidean plane. However, this property is not geometrical and has a strong discontinuity. This surprising property may be looked also clearly by the polar representation of a line.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)},$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle of OP ($P = (r, \theta)$ on the line) from the positive x axis. Then, if $\theta - \alpha = \pi/2$; that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics globally.

The distance d from the origin to the line determined by the two planes

$$\Pi_k : a_k x + b_k y + c_k z = 1, k = 1, 2,$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

If the two lines are coincident, then, of course, $d = 0$. However, if two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

9.3 Tangential lines and $\tan \frac{\pi}{2} = 0$

We looked the very fundamental and important formula $\tan \frac{\pi}{2} = 0$ in Section 6. In this subsection, for its importance we will furthermore see its various geometrical meanings.

We consider the high $\tan\theta \left(0 \le \theta \le \frac{\pi}{2}\right)$ that is given by the common point of two lines $y = (\tan\theta)x$ and $x = 1$ on the (x, y) plane. Then,

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$\tan\frac{\pi}{2} = 0,$$

by the division by zero. The result will show that, when $\theta = \pi/2$, two lines $y = (\tan\theta)x$ and $x = 1$ do not have a common point, because they are parallel in the usual sense. However, in the sense of the division by zero, parallel lines have the common point $(0, 0)$. Therefore, we can see the result $\tan \frac{\pi}{2} = 0$ following our new space idea.

We consider general lines represented by

$$ax + by + c = 0, \quad a'x + b'y + c' = 0.$$

The gradients are given by

$$k = -\frac{a}{b}, k' = -\frac{a'}{b'},$$

respectively. In particular, note that if $b = 0$, then $k = 0$, by the division by zero.

If $kk' = -1$, then the lines are orthogonal; that is,

$$\tan\frac{\pi}{2} = 0 = \pm\frac{k - k'}{1 + kk'},$$

which shows that the division by zero $1/0 = 0$ and orthogonality meets in a very good way.

Furthermore, even in the case of polar coordinates $x = r\cos\theta, y = r\sin\theta$, we can see the division by zero

$$\tan\frac{\pi}{2} = \frac{y}{0} = 0.$$

In particular, note the following fact.

From the expansion

$$\tan z = -\sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-(2\nu-1)\pi/2} + \frac{1}{(2\nu-1)\pi/2}\right),$$

$$\tan\frac{\pi}{2} = 0.$$

The division by zero may be looked even in the rotation of the coordinates.

We will consider a 2 dimensional curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a rotation defined by

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta.$$

Then, we write, by inserting these (x, y)

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0.$$

Then,

$$H = 0 \Longleftrightarrow \tan 2\theta = \frac{2h}{a-b}.$$

If $a = b$, then, by the division by zero,

$$\tan\frac{\pi}{2} = 0, \quad \theta = \frac{\pi}{4}.$$

For $h^2 > ab$, the equation

$$ax^2 + 2hxy + by^2 = 0$$

represents 2 lines and the angle θ made by two lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}.$$

If $h^2 - ab = 0$, then, of course, $\theta = 0$. If $a + b = 0$, then, by the division by zero, $\theta = \pi/2$ from $\tan\theta = 0$.

For a hyperbolic function

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \quad a, b > 0$$

the angle θ made by two asymptotic lines $y = \pm(b/a)x$ is given by

$$\tan\theta = \frac{2(b/a)}{1 - (b/a)^2}.$$

If $a = b$, then $\theta = \pi/2$ from $\tan\theta = 0$.

We recall that the equation passing two different points $(x_j, y_j); j = 1, 2)$ is given by

$$y = ax + b$$

$$= \frac{y_2 - y_1}{x_2 - x_1}x + y_1 - \frac{y_2 - y_1}{x_2 - x_1}x_1$$

$$= \frac{y_2 - y_1}{x_2 - x_1}x + y_2 - \frac{y_2 - y_1}{x_2 - x_1}x_2.$$

If $x_2 = x_1$, then, by the division by zero, we have the reasonable results that

$$a = 0, \quad b = y_1, y_2.$$

The line is parallel to the y axis and b is on the line $x = x_1$ (Hiroshi Michiwaki: 2020.10.26.19:05).

For a triangle OAB ($O(0,0), A(1,0), B(0,\tan\theta), \theta = \angle OAB$), we consider the inscribed circle

$$(x+r)^2 + y^2 = r^2$$

of A. Then, from

$$\frac{r}{r+1} = \tan\frac{\theta}{2},$$

we have

$$r = \frac{\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}.$$

For $\theta = \pi/2$, we have the reasonable result

$$r = \frac{1}{1-1} = 0.$$

For a line

$$x\cos\theta + y\sin\theta - p = 0$$

and for data (x_j, y_j), the minimum of $\sum_{j=1}^n d_j^2$ for the distance d_j of the line and the point (x_j, y_j) is attained for the case

$$\tan 2\theta = \frac{2\gamma_{xy}\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2},$$

where

$$\gamma_{xy} = \frac{n\sum_j x_j y_j - (\sum_j x_j)(\sum_j y_j)}{n^2\sigma_x\sigma_y}$$

and

$$\sigma_x = \frac{1}{n}\sqrt{n\sum_j x_j^2 - (\sum_j x_j)^2}.$$

If $\sigma_x^2 = \sigma_y^2$, then $\theta = \pi/4$ from $\tan 2\theta = 0$.

Here, of course, for

$$\mu_x = \frac{1}{n}\sum x_j, \quad \mu_y = \frac{1}{n}\sum y_j,$$

$$\sigma_x^2 = \frac{1}{n}\sum(x_j - \mu_x)^2, \quad \sigma_y^2 = \frac{1}{n}\sum(y_j - \mu_y)^2$$

and

$$\sigma_{xy}^2 = \frac{1}{n}\sum(x_j - \mu_x)(y_j - \mu_y),$$

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2}.$$

We consider the unit circle with its center at the origin on the (x, y) plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y = (\tan\theta)x \left(0 \leq \theta \leq \frac{\pi}{2}\right)$. Then, the distance R_θ between the common point and the common point of the tangential line and x-axis is given by

$$R_\theta = \tan\theta.$$

Then,

$$R_0 = \tan 0 = 0,$$

and

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$R_{\pi/2} = \tan\frac{\pi}{2} = 0.$$

This example shows also that by the stereographic projection mapping of the unit sphere with its center at the origin $(0, 0, 0)$ onto the plane, the north pole corresponds to the origin $(0, 0)$.

In this case, we consider the orthogonal circle C_{R_θ} with the unit circle through at the common point and the symmetric point with respect to the x-axis with its center $((\cos\theta)^{-1}, 0)$. Then, the circle C_{R_θ} is as follows:

C_{R_0} is the point $(1, 0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the y-axis and its curvature is also zero. Meanwhile, by the division by zero calculus, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

Note that from the expansion

$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty} (-1)^{\nu} \left(\frac{1}{z - (2\nu - 1)\pi/2} + \frac{2}{(2\nu - 1)\pi} \right),$$

$$\left(\frac{1}{\cos z}\right)\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi}\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{2\nu + 1} = 0.$$

The points $(\cos\theta, 0)$ and $((\cos\theta)^{-1}, 0)$ are the symmetric points with respect to the unit circle, and the origin corresponds to the origin.

In particular, the formal calculation

$$\sqrt{1 + R_{\pi/2}^2} = 1$$

is not good. The identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid formally for $\theta = \pi/2$.

This equation should be written as

$$\frac{\cos^2\theta}{\cos^2\theta} + \tan^2\theta = (\cos\theta)^{-2},$$

that is valid always.

Of course, as analytic functions, in the sense of the division by zero calculus, the identity is valid for $\theta = \pi/2$.

From the point at

$$x = \frac{1}{\cos\theta}$$

when we look the unit circle, we can see that the length $L(x)$ of the arc that we can see is given by

$$L(x) = 2\cos^{-1}\frac{1}{x}.$$

For $\theta = \pi/2$ that is for $x = 0$ we see that $L(x) = \pi$.

We fix $B(0, 1)$ and let $\angle ABO = \theta$ with $A(\tan\theta, 0)$. Let H be the point on the line BA such that two lines OH and AB are orthogonal. Then we see that

$$AH = \frac{\sin^2\theta}{\cos\theta}.$$

Note that for $\theta = \pi/2$, $AH = 0$.

Note that from the expansion

$$\frac{1}{\cos^2 z} = \sum_{\nu=-\infty}^{+\infty} \frac{1}{(z - (2\nu - 1)\pi/2)^2},$$

$$\left(\frac{1}{\cos^2 z}\right)\left(\frac{\pi}{2}\right) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

On the point $(p, q)(0 \le p, q \le 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with x-axis and y-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area S_p of the triangle formed by three points $(0, 0)$, $(1/p, 0)$ and $(0, 1/q)$ is given by

$$S_p = \frac{1}{2pq}.$$

Then,

$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$

however,

$$S_0 = 0$$

(H. Michiwaki: 2015.12.5.). We denote the point on the unit circle on the (x, y) plane with $(\cos\theta, \sin\theta)$ for the angle θ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the x-axis for the case $\theta \neq \pi/2$. Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5.).

The line through the points $(0, 1)$ and $(\cos\theta, \sin\theta)$ meets the x axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}.$$

Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

Note that

$$\frac{\cos x}{1 - \sin x} = -\frac{2}{x - \frac{x}{2}} + \frac{1}{6}(x - \frac{x}{2}) + \cdots.$$

From

$$\frac{1}{1 - \sin x} = \frac{2}{(x - \frac{x}{2})^2} + \frac{1}{6} + \frac{1}{120}(x - \frac{x}{2})^2 + \cdots,$$

it is

$$\frac{1}{6},$$

at $\pi/2$, meanwhile, note also that

$$\left[1 - \sin\left(\frac{\pi}{2}\right)\right]^{-1} = 0.$$

For a smooth curve $C : r = r(\theta), r \neq 0$, we consider the tangential line at P and a near point Q on the curve C. Let

H be the nearest point on the line OP, O is the pole of the coordinate and $\delta\theta$ is the angle for the line OP to the line OG. Then, we have

$$\tan\Theta := \lim_{\delta\theta\to 0}\frac{QH}{PH} = \frac{r(\theta)}{r'(\theta)}.$$

If $r'(\theta_0) = 0$, then $\tan\Theta = 0$ and $\Theta = \pi/2$, and the result is reasonable.

For the parabolic equation $y^2 = 4ax, a > 0$, at a point (x, y), the normal line shadow on the x-axis is given by

$$|yy'| = 2a.$$

At the origin, we have, from $y'(0) = 0$,

$$|yy'| = 0.$$

For the equation

$$x^m y^n = a^{m+n}, \quad a, m, n > 0,$$

let P be a point (x, y) on the curve. Let $T(x, x + (n/m)x)$ be the x cut of the tangential line of the curve and put $M(x, 0)$. Then, we have

$$TM : OM = -\frac{n}{m}.$$

This formula is valid for the cases $n = 0$ and $m = 0$, by the division by zero. Note that for both lines $x = a$ and $y = a$, the gradients are zero and $y' = 0$.

9.4 Two circles

We consider two circles with their radii a, b with their centers $(a, 0); a > 0$ and $(-b, 0); b > 0$, respectively. Then, the external common tangents $L_{a,b}$ (we assume that $a < b$ and that $L_{a,b}$ is not the y axis) has the common point with the x-axis at $(R_a, 0)$ which is given by, by fixing b

$$R_a = \frac{2ab}{b - a}. \tag{9.3}$$

We consider the circle C_{R_a} with its center at $(R_a, 0)$ with its radius R_a. Then,

$$a \to b \Longrightarrow R_a \to \infty,$$

however, when $a = b$, then we have $R_b = -2b$ by the division by zero calculus, from the identity

$$\frac{2ab}{b-a} = -2b - \frac{2b^2}{a-b}.$$

Meanwhile, when we consider (9.3) as

$$R_a = \frac{-1}{a-b} \cdot 2ab,$$

we have, for $a = b$, $R_b = 0$. It means that the circle C_{R_b} is the y axis with its curvature zero through the origin $(0, 0)$.

The above formulas will show a strong discontinuity for the change of a and b from $a = b$ (H. Okumura: 2015.10.29.).

We denote the circles S_j:

$$(x - a_j)^2 + (y - b_j)^2 = r_j^2.$$

Then, the common point (X, Y) of the co- and exterior tangential lines of the circles S_j for $j = 1, 2$, is given by

$$(X, Y) = \left(\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2}, \frac{r_1 b_2 - r_2 b_1}{r_1 - r_2} \right).$$

We will fix the circle S_2. Then, from the expansion

$$\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2} = \frac{r_2(a_2 - a_1)}{r_1 - r_2} + a_2 \tag{9.4}$$

for $r_1 = r_2$, by the division by zero, we have

$$(X, Y) = (a_2, b_2).$$

Meanwhile, when we consider (9.4) as

$$\frac{r_1a_2 - r_2a_1}{r_1 - r_2} = \frac{1}{r_1 - r_2} \cdot (r_1a_2 - r_2a_1),$$

we obtain that

$$(X, Y) = (0, 0),$$

that is reasonable. However, both cases, the results show a strong discontinuity.

9.5 Newton's method

The Newton's method is fundamental when we look for the solutions for some general equation $f(x) = 0$ numerically and practically. We will refer to its prototype case.

We will assume that a function $y = f(x)$ belongs to C^1 class. We consider the sequence $\{x_n\}$ for $n = 0, 1, 2, \ldots, n, \ldots$, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

When $f(x_n) = 0$, we have

$$x_{n+1} = x_n, \tag{9.5}$$

in the reasonable way. Even the case $f'(x_n) = 0$, we have also the reasonable result (9.5), by the division by zero.

9.6 Halley's method

As in the Newton's method, in order to look for the solution of the equation $f(x) = 0$, we consider the series

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{a[f'(x_n)]^2 - f(x_n)f''(x_n)}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left[1 - \frac{f(x_n)}{f'(x_n)} \frac{f''(x_n)}{2f'(x_n)}\right]^{-1}.$$

If $f(x_n) = 0$, the processes stop and there is no problem. Even the case $f'(x_n) = 0$, the situation is similar.

9.7 Cauchy's mean value theorem

For the Cauchy mean value theorem; that is, for $f, g \in Differ(a, b)$, differentiable, and $\in C^0[a, b]$, continuous and if $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, then there exists $\xi \in (a, b)$ satisfying that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)},$$

we do not need the assumptions $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, by the division by zero. Indeed, if $g(a) = g(b)$, then, by the Rolle theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$. Then, both terms are zero and the equality is valid.

For $f, g \in C^2[a, b]$, there exists $\xi \in (a, b)$ satisfying

$$\frac{f(b) - f(a) - (b - a)f'(a)}{g(b) - g(a) - (b - a)g'(a)} = \frac{f''(a)}{g''(a)}.$$

Here, we do not need the assumption

$$g(b) - g(a) - (b - a)g'(a) \neq 0,$$

by the division by zero.

For a function $f \in C^2$ satisfying $f(\alpha) = 0, f'(\alpha) > 0$ and for a small k, for the solution $x = \alpha + \delta$, we obtain

$$\delta \sim \frac{-f'(\alpha) + \sqrt{f'(\alpha)^2 + 2kf''(\alpha)}}{f''(\alpha)}.$$

If here $f''(\alpha) = 0$, then, the team should be replaced by

$$\frac{k}{f'(\alpha)}$$

([116], 71 page.). This modification can be derived by the division by zero calculus, because in

$$\frac{\sqrt{A+2kx}}{x},$$

for $x = 0$

$$\frac{k}{\sqrt{A}}.$$

9.8 Length of tangential lines

We will consider the inversion $A(1/x, 0)$ of a point $X(x, 0), 0 < x < 1$ with respect to the unit circle with its center the origin. Then the length $T(x)$ of the tangential line AB ($B(x, \sqrt{1-x^2})$) is given by

$$T(x) = \frac{1}{x}\sqrt{1-x^2}.$$

For $x = 0$, by the division by zero calculus, we have

$$T(0) = 0$$

that was considered as $+\infty$.

We will consider a function $y = f(x)$ of C^1 class on the real line. We consider the tangential line through $(x, f(x))$

$$Y = f'(x)(X - x) + f(x).$$

Then, the length (or distance) $d(x)$ between the point $(x, f(x))$ and $\left(x - \frac{f(x)}{f'(x)}, 0\right)$ is given by, for $f'(x) \neq 0$

$$d(x) = |f(x)|\sqrt{1 + \frac{1}{f'(x)^2}}.$$

How will be the case $f'(x^*) = 0$? Then, the division by zero shows that

$$d(x^*) = |f(x^*)|.$$

Meanwhile, the x axis point $(X_t, 0)$ of the tangential line at (x, y) and y axis point $(0, Y_n)$ of the normal line at (x, y) are given by

$$X_t = x - \frac{f(x)}{f'(x)}$$

and

$$Y_n = y + \frac{x}{f'(x)},$$

respectively. Then, if $f'(x) = 0$, we obtain the reasonable results:

$$X_t = x, \quad Y_n = y.$$

For a smooth curve $r = f(\theta)$ with polar coordinate with respect to the origin O, on a point $P(r, \theta)$, we consider the line that is orthogonal to the line OP, and let T and N be the common points with the line and the tangential line and the normal line at the point P, respectively. Then, we have

$$\overline{OT} = \frac{r^2}{r'},$$

$$\overline{ON} = r',$$

and

$$\overline{PT} = \frac{r}{r'}\sqrt{r^2 + r'^2}.$$

Consider the circle case $r = a$, then we have $r' = 0$. Then, we see that the above three formulas are zero.

On a fixed circle

$$(x - r)^2 + (y - r)^2 = r^2, r > 0,$$

we consider the tangential line with its common points A and B with the x positive line and y positive line, respectively. For the triangle ABC, C is the origin, we have the identity

$$\frac{r}{\sin\frac{A}{2}} = \sqrt{2}c \sin\frac{B}{2}.$$

If $A = 0$, then we have the identity

$$\frac{r}{\sin 0} = \sqrt{2} \cdot 0 \cdot \sin \frac{\pi}{4} = 0.$$

On the same situation, we consider the tangential line at the point $T = (\sqrt{r^2 - h^2}, h), h > 0$ and let L be the length $L = TA$. Then, we have

$$L = \frac{rh}{\sqrt{r^2 - h^2}}.$$

For $h = r$, by the division by zero calculus, we have that $L = 0$ (V. V. Puha: 2018.7.3.06:19).

9.9 Curvature and center of curvature

We will assume that a function $y = f(x)$ is of class C^2. Then, the curvature radius ρ and the center $O(x, y)$ of the curvature at point $(x, f(x))$ are given by

$$\rho(x, y) = \frac{(1 + (y')^2)^{3/2}}{y''}$$

and

$$O(x, y) = \left(x - \frac{1 + (y')^2}{y''} y', y + \frac{1 + (y')^2}{y''} \right),$$

respectively. Then, if $y'' = 0$, we have the results

$$\rho(x, y) = 0$$

and

$$O(x, y) = (x, y),$$

by the division by zero. They are reasonable.

We will consider a curve $\mathbf{r} = \mathbf{r}(s), s = s(t)$ of class C^2. Then,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \mathbf{t} = \frac{d\mathbf{r}(\mathbf{s})}{ds}, v = \frac{ds}{dt}, \frac{d\mathbf{t}(\mathbf{s})}{ds} = \frac{1}{\rho}\mathbf{n},$$

by the principal normal unit vector $\mathbf{n}$. Then, we see that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{t} + \frac{v^2}{\rho}\mathbf{n}.$$

If $\rho(s_0) = 0$: (consider a line case), then

$$\mathbf{a}(s_0) = \left[\frac{dv}{dt}\mathbf{t}\right]_{s=s_0}$$

and

$$\left[\frac{v^2}{\rho}\right]_{s=s_0} = \infty$$

will be funny. It will be the zero.

Sciacci's theorem: *When we set $h = v/\rho$, we have, for the distance p from the origin to the tangential line at P and the radial component and tangential component of $\mathbf{a}$ at P*

$$\mathbf{a}_r = \frac{h^2 r\rho}{p^2}$$

and

$$\mathbf{a}_t = \frac{h}{p^2}\frac{dh}{ds}.$$

For the curvature zero, of course, $\mathbf{a}_r = 0$.

9.10 $n = 2, 1, 0$ regular polygons inscribed in a disc

We consider n regular polygons inscribed in a fixed disc with its radius a. Then we note that their area S_n and the length L_n of the sum of the sides are given by

$$S_n = \frac{na^2}{2}\sin\frac{2\pi}{n}$$

and

$$L_n = 2na\sin\frac{\pi}{n},$$

respectively. For $n \geq 3$, the results are clear.

For $n = 2$, we will consider two diameters that are the same. We can consider it as a generalized regular polygon inscribed in the disc as a degenerate case. Then, $S_2 = 0$ and $L_2 = 4a$, and the general formulas are valid.

Next, we will consider the case $n = 1$. Then the corresponding regular polygon is a just diameter of the disc. Then, $S_1 = 0$ and $L_1 = 0$ that will mean that any regular polygon inscribed in the disc may not be formed and so its area and length of the side are zero.

For $n = 1$ triangle, if 1 means one side, then we can consider as in the above, however, if we consider 1 as one vertex, the above situation may be considered as one point on the circle which will mean $S_1 = L_1 = 0$.

Now we will consider the case $n = 0$. Then, by the division by zero calculus, we obtain that $S_0 = \pi a^2$ and $L_0 = 2\pi a$. Note that they are the area and the length of the disc. How to understand the results? Imagine contrary n tending to infinity, then the corresponding regular polygons inscribed in the disc tend to the disc. Recall our new idea that the point at infinity is represented by 0. Therefore, the results say that $n = 0$ regular polygons are $n = \infty$ regular polygons inscribed in the disc in a sense and they are the disc. This is our interpretation of the theorem:

Theorem 9.1 $n = 0$ *regular polygons inscribed in a disc are the whole disc.*

Note that the radius r_n of the inscribed circle in the n regular polygon

$$r_n = a \cos \frac{\pi}{n}.$$

For $n \to \infty$, $r_n \to a$, however, for $n = 0$, $r_0 = a$ (H. Okumura: 2019.2.1.).

In addition, note that each inner angle A_n of a general n regular polygon inscribed in a fixed disc with its radius a is given by

$$A_n = \left(1 - \frac{2}{n}\right)\pi. \tag{9.6}$$

The circumstances are similar for n regular polygons circumscribed in the disc, because the corresponding data are given by

$$S_n = na^2 \tan\frac{\pi}{n}$$

and

$$L_n = 2na \tan\frac{\pi}{n},$$

and (9.6), respectively.

In connection with the interesting example, we will refer to another example.

On a disc with radius R, we consider n regular polygon inscribed the disc $A_1A_2 \cdots A_n$. Let r_j be the radius of the inscribed circle in the triangle $A_jA_{j+1}A_{j+2}$. Then, we have

$$I_n = r_1 + r_2 + \cdots + r_{n-2} = 2R\left(1 - n\sin^2\frac{\pi}{2n}\right).$$

For $n \to \infty$, we have

$$\lim_{n\to\infty} I_n = 2R.$$

Meanwhile, by the division by zero calculus, we have

$$I_0 = 2R.$$

9.11 Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{|\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2}\right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k, 0)$ with its radius r. Then, we have

$$r = k \sin \alpha.$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately

$$[r]_{r=\infty} = 0.$$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

9.12 H. Okumura's example

The suprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a circle C_a of fixed radius $a > 0$ inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive y axis is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

On the horn torus models of Puha and Däumler, the example in Subsection 9.11 is clear.

Meanwhile, on the Okumura example, note that the series of discs tending to the point at infinity converges to the crucial point of the horm torus on the upper part. However, the disc with its center at the origin, of course, is mapped to the lower part of the horn torus. Therefore, we see the surprising result:

Conclusion: The Okumura's disc series can beyond the crucial point of Däumler-Puha's horn torus models for the Riemann sphere.

These three subsections were taken from [57].

In connection with the examples in Subsections 9.11 and 9.12, we note the following example.

We consider the circles, for fixed $r > 0$

$$(x-a)^2 + y^2 = r^2.$$

For the case $a = 0$, we have

$$x^2 + y^2 = r^2.$$

Meanwhile, from

$$\frac{x^2}{a} - 2x + a + \frac{y^2}{a} = \frac{r^2}{a},$$

by the division by zero, we have that

$$x = 0.$$

Similarly we consider the circles

$$(x-r)^2 + (y-r)^2 = r^2.$$

First, for $r = 0$, the cirlcle is a point $(0, 0)$, and from the identity

$$\frac{x^2 + y^2}{r^2} - \frac{2(x + y)}{r} + \frac{r^2}{r^2} = 0$$

and from $r = 0$, we have the nonsense result. However, from the identity

$$\frac{x^2 + y^2}{r} - 2(x + y) + r = 0$$

and from $r = 0$, we have the interesting result

$$y = -x.$$

9.13 Interpretation by analytic geometry

The results in Subsection 9.1 may be derived beautifully by analytic geometry and matrix theory.

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2, 3.$$

The area S of the triangle surrounded by these lines is given by

$$S = \pm \frac{1}{2} \cdot \frac{\triangle^2}{D_1 D_2 D_3},$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and D_k is the co-factor of $\triangle$ with respect to c_k. $D_k = 0$ if and only if the corresponding lines are parallel. $\triangle = 0$ if and only if the three lines are parallel or they have a common point. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

Similarly we write lines by

$$M_k : a_{k1} x + a_{k2} y + a_{3k} = 0, k = 1, 2, 3.$$

The area S of the triangle surrounded by these lines is given by

$$S = \frac{1}{A_{11}A_{22}A_{33}} \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

where A_{kj} is the co-factor of a_{kj} with respect to the matrix $[a_{kj}]$. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{9.7}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero. In this case, the circle is a line (degenerated).

The center of the circle (9.7) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a} \right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

On the complex z plane, a circle containing a line is represented by the equation

$$az\overline{z} + \overline{\alpha}z + \alpha\overline{z} + c = 0,$$

for a, c : real and $ac \leq \overline{\alpha}\alpha$. Then the center and the radius are given by

$$-\frac{\alpha}{a}$$

and

$$\frac{\sqrt{\alpha\overline{\alpha} - ac}}{a},$$

respectively. If $a = 0$, then it is a line with center $(0, 0)$ with radius 0, by the division by zero. The curvature of the line is, of course, zero, by the division by zero.

H. Okumura ([80]) considers some beautiful circles with their radius $d_n = a/n, n = 0, 1, 2, 3, ...a > 0$ fixed and gives the natural and beautiful interpretation for $d_0 = 0$.

We consider the functions

$$S_j(x, y) = a_j(x^2 + y^2) + 2g_j x + 2f_j y + c_j.$$

The distance d of the centers of the circles $S_1(x, y) = 0$ and $S_2(x, y) = 0$ is given by

$$d^2 = \frac{g_1^2 + f_1^2}{a_1^2} - 2\frac{g_1 g_2 + f_1 f_2}{a_1 a_2} + \frac{g_2^2 + f_2^2}{a_2^2}.$$

If $a_1 = 0$, then by the division by zero

$$d^2 = \frac{g_2^2 + f_2^2}{a_2^2}.$$

Then, $S_1(x, y) = 0$ is a line and its center is the origin $(0, 0)$. Therefore, the result is very reasonable.

The distance d between two lines given by

$$\frac{x - a_j}{L_1} = \frac{y - b_j}{M_j} = \frac{z - c_j}{N_j}, \quad j = 1, 2,$$

is given by

$$d = \frac{\begin{vmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix}}{\sqrt{(M_l N_2 - M_2 N_1)^2 + (N_l L_2 - N_2 L_1)^2 + (L_l M_2 - L_2 M_1)^2}}.$$

If two lines are parallel, then we have $d = 0$.

9.14 Interpretation with volumes

We write four planes by

$$\pi_k : a_k x + b_k y + c_k z + d_k = 0, k = 1, 2, 3, 4.$$

The volume V of the tetrahedron surrounded by these planes is given by

$$V = \pm\frac{1}{6} \cdot \frac{\triangle^2}{D_1 D_2 D_3 D_4},$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and D_k is the co-factor of $\triangle$ with respect to d_k. $D_k = 0$ if and only if two planes of the corresponding three planes are parallel. $\triangle = 0$ if and only if the four planes π_k contain four lines L_k (for each k, respectively) that are parallel or have a common line. We can see that the degeneracy of the tetrahedron may be considered by $V = 0$ beautifully, by the division by zero.

This subsection was taken from [57].

9.15 Interpretation for minus area

We first recall the typical example for the area of a triangle. The area S of the triangle $P_1P_2P_3$ with $P_j(x_j, y_j), j = 1, 2, 3$ is given by

$$S = \pm\frac{1}{2} \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For the sigh, when we consider P_1, P_2, P_3 for the direction of the triangle, + and in the converse (reverse) direction, −. The property shows a beautiful relation of geometry and algebra. We can see many and many examples as a beautiful property.

Here, we will give a reason why such a relation exists. In this concrete case, we can say surprisingly that the minus area shows the area of the outside of the triangle **in a new sense** that is derived by the division by zero.

Since a general result and a special case are in the same situation, we willl state our conclusion in the special case.

We consider a large disc containing the triangle $P_1P_2P_3$ with radius R with center at the origin. Then, the area $S(R)$ of $\{$ $\{x^2 + y^2 < R^2\} \setminus \triangle P_1P_2P_3\}$ is given by

$$S(R) = \pi R^2 - S.$$

Of course,

$$\lim_{R \to \infty} S(R) = +\infty,$$

However, by the division by zero, for $R = \infty$, we obtain

$$S(\infty) = -S,$$

that means the area of the outside of the triangle.

This subsection was taken from [107].

10 APPLICATIONS TO WASAN GEOMETRY

For the sake of the great contributions to Wasan geometry by H. Okumura, we found new interesting results as applications of the division by zero calculus. We will introduce typical results, however, the results and their impacts will create some new fields in mathematics.

10.1 Circle and line

We will consider a fixed circle $x^2 + (y - b)^2 = b^2, b > 0$. For a taching circle with this circle and the x axis is represented by

$$(x - 2\sqrt{ab})^2 + (y - a)^2 = a^2.$$

Then, we have

$$\frac{x^2 + y^2}{\sqrt{a}} - 4\sqrt{b}x = 2\sqrt{a}(y - 2b)$$

and

$$\frac{x^2 + y^2}{a} - 4\sqrt{\frac{b}{a}}x = 2(y - 2b).$$

Then, by the division by zero, we have the reasonable results the origin, that is the point circle of the origin, the y axis and the line $y = 2b$, respectively (H. Okumura: 2017.10.13.).

We consider a fixed circle: $(x - a)^2 + y^2 = a^2$. We will consider the circles tauching with both the y axis and the circle. Those circles may be represented with a parameter z as follows:

$$C_z : \left(x - \frac{a}{z^2}\right)^2 + \left(y - \frac{2a}{z}\right)^2 = \left(\frac{a}{z^2}\right)^2.$$

Then, when z tends to infinity, we have the point circle of the origin.

Meanwhile, from the representation

$$(x^2 + y^2) - \frac{2ax}{z^2} - \frac{4ay}{z} + \frac{4a^2}{z^2} = 0,$$

for $z = 0$, we have $x = y = 0$; the point circle, by the division by zero calculus.

From the expression

$$z(x^2 + y^2) - \frac{2ax}{z} - 4ay + \frac{4a^2}{z} = 0,$$

we have the x axis.

From the expression

$$z^2(x^2 + y^2) - 2ax - 4ayz + 4a^2 = 0,$$

we have $x = 2a$.

Furthermore, for the relation of the two circles C_z and C_w that are tauching each other, we have the beautiful relation $|z - w| = 1$ ([77]), (H. Okumura: 2020.1.1.16:09).

See [68, 78] for furthermore results.

10.2 Three externally touching circles

For real numbers z, and $a, b > 0$, the point $(0, 2\sqrt{ab}/z)$ is denoted by V_z. H. Okumura and M. Watanabe gave the theorem in [82]:

Theorem 7. *The circle touching the circle α: $(x-a)^2+y^2 = a^2$ and the circle β: $(x + b)^2 + y^2 = b^2$ at points different from the origin O and passing through $V_{z\pm1}$ is represented by*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{10.1}$$

for a real number $z \neq \pm 1$.

The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{10.2}$$

Anyhow the authors give the exact representation with a parameter of the general circles touching with two circles touching each other. The common external tangents may be looked as circles touching for the general circles (as we know we can consider circles and lines as same ones in complex analysis or with the stereographic projection or in the representation of a circle by the equation), however, they stated in the proof of the theorem that the common external tangents are obtained by the limiting $z \to \pm 1$. However, its logic will have a delicate problem.

Following our concept of the division by zero calculus, we will consider the case $z^2 = 1$ for the singular points in the general parametric representation of the touching circles.

10.2.1 Results

First, for $z = 1$ and $z = -1$, respectively by the division by zero calculus, we have from (10.1), surprisingly

$$x^2 + \frac{b-a}{2}x + y^2 \mp \sqrt{ab}y - ab = 0, \tag{10.3}$$

respectively.

Secondly, multiplying (10.1) by $(z^2 - 1)$, we immediately obtain surprisingly (10.2) for $z = 1$ and $z = -1$, respectively by the division by zero calculus.

In the usual way, when we consider the limiting $z \to \infty$ for (10.1), we obtain the trivial result of the point circle of the origin. However, the result may be obtained by the division by zero calculus at $w = 0$ by setting $w = 1/z$.

10.2.2 On the circle appeared

The circle (10.3) meets the circle α in two points

$$P_a\left(2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{a}{b}}\right), \quad Q_a\left(\frac{2ab}{9a+b}, -\frac{6a\sqrt{ab}}{9a+b}\right),$$

where $r_{\mathrm{A}} = ab/(a+b)$. Also it meets β in points

$$P_b\left(-2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{b}{a}}\right), \quad Q_b\left(\frac{-2ab}{a+9b}, -\frac{6b\sqrt{ab}}{a+9b}\right).$$

The line P_aP_b is the common tangential of two circles α and β on the upper half plane. The lines P_aQ_a and P_bQ_b intersect at the point $R : \left(0, -\sqrt{ab}\right)$, which lies on the remaining tangentials of α and β. Furthermore, the circle (10.3) is orthogonal to the circle with center R passing through the origin.

The source of this subsection is [86].

10.3 The Descartes circle theorem

We recall the famous and beautiful theorem ([46, 114]):

Theorem (Descartes). *Let C_i $(i = 1, 2, 3)$ be circles touching to each other of radii r_i. If a circle C_4 touches the three circles, then its radius r_4 is given by*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}}. \tag{10.4}$$

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and with many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

We would like to consider all cases for the Descartes theorem for lines and point circles, step by step.

10.3.1 One line and two circles case

We consider the case in which the circle C_3 is one of the external common tangents of the circles C_1 and C_2. This is a typical case in this paper. We assume that $r_1 \geq r_2$. We now have $r_3 = 0$ in (10.4). Hence

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{0} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 \cdot 0} + \frac{1}{0 \cdot r_1}} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1 r_2}}.$$

This implies

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

in the plus sign case. The circle C_4 is the incircle of the curvilinear triangle made by C_1, C_2 and C_3. In the minus sign case we have

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_2}} - \frac{1}{\sqrt{r_1}}.$$

In this case C_2 is the incircle of the curvilinear triangle made by the other three.

Of course, the result is known. The result was also well-known in Wasan geometry [125] with the Descartes circle theorem itself.

10.3.2 Two lines and one circle case

In this case, the two lines have to be parallel, and so, this case is trivial, because then other two circles are the same size circles, by the division by zero $1/0 = 0$.

10.3.3 One point circle and two circles case

This case is another typical case for the theorem. Intuitively, for $r_3 = 0$, the circle C_3 is the common point of the circles C_1 and C_2. Then, there does not exist any touching circle of the three circles $C_j; j = 1, 2, 3$.

For the point circle C_3, we will consider it by limiting of circles attaching to the circles C_1 and C_2 to the common point. Then, we will examine the circles C_4 and the Descartes theorem.

In Theorem 7, by setting $z = 1/w$, we will consider the case $w = 0$; that is, the case $z = \infty$ in the classical sense; that is, the circle C_3 is reduced to the origin.

We look for the circles C_4 attaching with three circles $C_j; j = 1, 2, 3$. We set

$$C_4 : (x - x_4)^2 + (y - y_4)^2 = r_4^2. \tag{10.5}$$

Then, from the touching property we obtain:

$$x_4 = \frac{r_1 r_2 (r_2 - r_1) w^2}{D},$$

$$y_4 = \frac{2 r_1 r_2 \left(\sqrt{r_1 r_2} + (r_1 + r_2) w\right) w}{D}$$

and

$$r_4 = \frac{r_1 r_2 (r_1 + r_2) w^2}{D},$$

where

$$D = r_1 r_2 + 2\sqrt{r_1 r_2}(r_1 + r_2) w + (r_1^2 + r_1 r_2 + r_2^2) w^2.$$

By inserting these values to (10.5), we obtain

$$f_0 + f_1 w + f_2 w^2 = 0,$$

where

$$f_0 = r_1 r_2 (x^2 + y^2),$$

$$f_1 = 2\sqrt{r_1r_2}\big((r_1 + r_2)(x^2 + y^2) - 2r_1r_2y\big)$$

and

$$f_2 = (r_1^2+r_1r_2+r_2^2)(x^2+y^2)+2r_1r_2(r_2-r_1)x-4(r_1+r_2)y+4r_1^2r_2^2.$$

By using the division by zero calculus for $w = 0$, we obtain, for the first, for $w = 0$, the second by setting $w = 0$ after dividing by w and for the third case, by setting $w = 0$ after dividing by w^2,

$$x^2 + y^2 = 0, \tag{10.6}$$

$$(r_1 + r_2)(x^2 + y^2) - 2r_1r_2y = 0 \tag{10.7}$$

and

$$(r_1^2 + r_1r_2 + r_2^2)(x^2 + y^2) + 2r_1r_2(r_2 - r_1)x \tag{10.8}$$
$$-4r_1r_2(r_1 + r_2)y + 4r_1^2r_2^2 = 0.$$

Note that (10.7) is the circle with the radius

$$\frac{r_1r_2}{r_1 + r_2} \tag{10.9}$$

and (10.8) is the circle whose radius is

$$\frac{r_1r_2(r_1 + r_2)}{r_1^2 + r_1r_2 + r_2^2}.$$

When the circle C_3 is reduced to the origin, of course, the inscribed circle C_4 is reduced to the origin, then the Descartes theorem is not valid. However, by the division by zero calculus, then the origin of C_4 is changed suddenly for the cases (10.6), (10.7) and (10.8), and for the circle (10.7), the Descartes theorem is valid for $r_3 = 0$, surprisingly.

Indeed, in (10.4) we set $\xi = \sqrt{r_3}$, then (10.4) is as follows:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\xi^2} \pm 2\frac{1}{\xi}\sqrt{\frac{\xi^2}{r_1r_2} + \left(\frac{1}{r_1} + \frac{1}{r_2}\right)},$$

and so, by the division by zero calculus at $\xi = 0$, we have

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2}$$

which is (10.9). Note, in particular, that the division by zero calculus may be applied in many ways and so, for the results obtained should be examined for some meanings. This circle (10.7) may be looked a circle touching the origin and two circles C_1 and C_2, because by the division by zero calculus

$$\tan \frac{\pi}{2} = 0,$$

that is a popular property.

Meanwhile, the circle (10.8) is the attaching circle with the circles C_1, C_2 and the beautiful circle with its center $((r_2 - r_1), 0)$ with its radius $r_1 + r_2$. Each of the areas surrounded by three circles C_1, C_2 and the circle of radius $r_1 + r_2$ is called an arbelos, and the circle (10.7) is the famous Bankoff circle of the arbelos. For $r_3 = -(r_1 + r_2)$, from the Descartes identity (10.4), we have (10.4). That is, when we consider that the circle C_3 is changed to the circle with its center $((r_2 - r_1), 0)$ with its radius $r_1 + r_2$, the Descartes identity holds. Here, the minus sign shows that the circles C_1 and C_2 touch C_3 internally from the inside of C_3.

10.3.4 Two point circles and one circle case

This case is trivial, because, the exterior touching circle is coincident with one circle.

10.3.5 Three points case and three lines case

In these cases we have $r_j = 0, j = 1, 2, 3$ and the formula (10.4) shows that $r_4 = 0$. This statement is trivial in the general sense.

As the solution of the simplest equation

$$ax = b, \tag{10.10}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (10.10) is impossible. The zero will represent some **impossibility**.

In the Descartes theorem, three lines and three points cases, we can understand that the attaching circle does not exist, or it is the point and so the Descartes theorem is valid.

This subsection is based on the paper [84].

10.4 Circles and a chord

We recall the following result of the old Japanese geometry [124, 114, 82]:

Lemma 10. *Assume that the circle C with its radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with their radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

We are interesting in the limit case $r_1 = 0$ or $r_2 = 0$. In order to see the backgound of the lemma, we will see its simple proof.

The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i $(i = 1, 2)$, the line parallel to t through the center of C, and the normal dropped on t from the center of C_i, we have

$$|\sqrt{(r - r_1)^2 - (h + r_1 - r)^2} \pm \sqrt{(r - r_2)^2 - (h + r_2 - r)^2}|$$

$$= 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the tangency points is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation.

10.4.1 Results

We introduce the coordinates in the following way. The bottom of the circle C is the origin and tangential line at the origin of the circle C is the x axis and the y axis is given as in the center of the circle C is $(0, r)$. We denote the centers of the circles $C_j; j = 1, 2$ by (x_j, y_j), then we have

$$y_1 = h + r_1, \quad y_2 = h + r_2.$$

Then, from the attaching conditions, we obtain the three equations:

$$(x_2 - x_1)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2,$$
$$x_1^2 + (h - r + r_1)^2 = (r - r_1)^2$$

and

$$x_2^2 + (h - r + r_2)^2 = (r - r_2)^2.$$

Solving the equations for x_1, x_2 and r_2, we get four sets of the solutions. Let $h = 2r_3$, $v = r - r_1 - r_3$. Then two sets are given by

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \pm 2\frac{r_1\sqrt{r r_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

The other two sets are

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \mp 2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

We now consider the solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{g_0 + g_1 r_1 + g_2 r_1^2 + g_3}{(r_1 + r_3)^2},$$

where

$$g_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$

$$g_1 = 2r_3((x - \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$

$$g_2 = (x - 2\sqrt{rr_3})^2 + y^2 - 2r_3 y,$$

and

$$g_3 = 4r_3\sqrt{v}(r_1(\sqrt{r}y - \sqrt{r_3}x) - r_3\sqrt{r_3}x).$$

We now consider another solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= -2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{k_0 + k_1 r_1 + k_2 r_1^2 + k_3}{(r_1 + r_3)^2},$$

where

$$k_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$

$$k_1 = 2r_3((x + \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$

$$k_2 = (x + 2\sqrt{rr_3})^2 + y^2 - 2r_3 y,$$

and

$$k_3 = -4r_3\sqrt{v}(r_1(\sqrt{r}y + \sqrt{r_3}x) + r_3\sqrt{r_3}x).$$

We thus see that the circle C_2 is represented by

$$(g_0 + g_3) + g_1 r_1 + g_2 r_1^2 = 0$$

and

$$(k_0 + k_3) + k_1 r_1 + k_2 r_1^2 = 0.$$

For the symmetry, we consider only the above case. We obtain the division by zero calculus, first by setting $r_1 = 0$, the next by setting $r_1 = 0$ after dividing by r_1 and the last by setting $r_1 = 0$ after dividing by r_1^2,

$$g_0 + g_3 = 0,$$

$$g_1 = 0,$$

and

$$g_2 = 0.$$

That is,

$$\left(x - \sqrt{2rh - h^2}\right)^2 + (y - h)^2 = 0,$$

$$\left(x - \sqrt{\frac{rh}{2}}\right)^2 + \left(y - \left(r + \frac{3h}{4}\right)\right)^2 = r^2 + \frac{9}{16}h^2,$$

and

$$\left(x - \sqrt{2rh}\right)^2 + \left(y - \frac{h}{2}\right)^2 = \left(\frac{h}{2}\right)^2.$$

The first equation represents the point $(\sqrt{2rh - h^2}, h)$ of the intersection of the circle C and the chord t. The second equation expresses the circle with the center and radius given the equation. The third equation expresses the circle touching C externally, the x-axis and the extended chord t. The last two circles are orthogonal to the circle with center at origin passing through the points of intersection of C and t.

Now for the beautiful identity in the lemma, for $r_1 = 0$, we have, by the division by zero,

$$\frac{1}{0} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{0 \cdot r_2 h}}$$

and

$$r_2 = -\frac{h}{2}.$$

Here, the minus sign will mean that the second circle is attaching with the circle C in the outside of the circle C; that is, we can consider that when the circle C_1 is reduced to the point $(\sqrt{2rh - h^2}, h)$, then the circle C_2 is suddenly changed to the second circle and the beautiful identity is still valid. Note, in particular, the second circle is attaching with the circle C and the chord t.

Meanwhile, for the curious second circle, we do not know its property, however, we know curiously that it is orthogonal with the circle with the center at the origin and with radius $\sqrt{2rh}$ passing through the points $(\pm\sqrt{2rh - h^2}, h)$.

This subsection is based on the paper [86].

We can find more many division by zero calculus examples in geometry. See, for example, [66, 67]. In particular, see the beautiful Figure 14 in H. Okumura ([66]). There, the ratio of 6 same size circles and the large circle are constant and when the radius is zero, we see a beautiful result.

Further interesting results, see H. Okumura ([73, 75, 79]).

10.5 Okumura's Laurent expansion and division by zero

First, we recall Okumura's results from ([81]). Let C be a point on the segment AB such that $|BC| = 2a$ and $|CA| = 2b$. We consider the three circles α, β and γ with diameters BC, CA and AB, respectively. We use a rectangular coordinates system with origin C such that the point B has coordinates $(2a, 0)$. We call the line AB the baseline. Let $c = a + b$ and $d = \sqrt{ab}/c$. Then, we have:

A circle γ_z touches the circles α and β if and only if it has radius r_z^γ and center of coordinates (x_z^γ, y_z^γ) given by

$$r_z^\gamma = |q_z^\gamma| \text{ and } (x_z^\gamma, y_z^\gamma) = \left(\frac{b-a}{c} q_z^\gamma, 2zq_z^\gamma\right), \text{ where } q_z^\gamma = \frac{abc}{c^2 z^2 - ab}$$

for a real number $z \neq \pm d$.

The circle γ_z touches α and β internally (resp. externally) if and only if $|z| < d$ (resp. $|z| > d$). The external common tangents of α and β have following equations:

$$(a - b)x \mp 2\sqrt{ab}y + 2ab = 0,$$

which are denoted by $\gamma_{\pm d}$.

The distance between the center of the circle γ_z and the baseline equals $2|z|r_z^\gamma$.

The ratio of the distance from the center of γ_z to the perpendicular to the baseline at C to the radius of γ_z is constant and equals to $|a - b|/c$ for $z \neq \pm d$.

Let $g_z(x, y) = (x-x_z^\gamma)^2+(y-y_z^\gamma)^2-(r_z^\gamma)^2$. Then $g_z(x, y) = 0$ is an equation of the circle γ_z for $z \neq \pm d$. Let

$$g_z(x, y) = \cdots+C_{-2}(z-d)^{-2}+C_{-1}(z-d)^{-1}+C_0+C_1(z-d)+\cdots$$

be the Laurent expansion of $g_z(x, y)$ around $z = d$, then we have

$$\cdots = C_{-4} = C_{-3} = C_{-2} = 0,$$

$$C_{-1} = d((a-b)x - 2\sqrt{ab}y + 2ab),$$

$$C_0 = \left(x - \frac{a-b}{4}\right)^2 + \left(y - \frac{\sqrt{ab}}{2}\right)^2 - \left(\frac{\sqrt{a^2+18ab+b^2}}{4}\right)^2,$$

$$C_n = -\frac{1}{2}\left(\frac{-1}{2d}\right)^n ((a-b)x + 2\sqrt{ab}y + 2ab),\ for\ n = 1, 2, 3, \cdots.$$

Therefore $C_{-1} = 0$ gives an equation of the line γ_d. Also $C_n = 0$ gives an equation of the line γ_{-d} for $n = 1, 2, 3, \cdots$.

Let ε be the circle given by the equation $C_0 = 0$. Then, it has the following beautiful properties that were given in [85]:

(i) The points, where γ_d touches α and β, lie on ε.
(ii) The radical center of the three circles α, β and ε has coordinates $(0, -\sqrt{ab})$, and lies on the line γ_{-d}.
(iii) The radical axis of the circles ε and γ passes though the points of coordinates $(0, 3\sqrt{ab})$ and $(2ab/(b-a), 0)$, where the latter coincides with the point of intersection of γ_d and γ_{-d}.
The y-axis meets γ and $\gamma_{\pm d}$ in the points of coordinates $(0, \pm 2\sqrt{ab})$ and $(0, \pm\sqrt{ab})$, respectively. Hence the six points, where the y-axis meets γ, $\gamma_{\pm d}$, the baseline, the radical axis of γ and ε, are evenly spaced. Reflecting the figure in the baseline, we also get similar results for the Laurent expansion of $g_z(x, y)$ around $z = -d$.

Meanwhile, for the Laurent expansion of $g_z(x, y)$ around $z = 0$, we have:

$$\cdots = C_{-3} = C_{-2} = C_{-1} = 0,$$

$$C_0 = (x-2a)(x+2b) + y^2 = g_0(x, y),$$

$$C_n = \frac{4(a+b)^n}{(ab)^{(n-1)/2}} y; \quad n = 1, 3, 5, \cdots,$$

and

$$C_n = -\frac{2(a+b)^n}{(ab)^{(n/2)}}(a-b)\left(x - \frac{2ab}{b-a}\right); \quad n = 2, 4, 6, \cdots.$$

Therefore $C_0 = 0$ is an equation of the circle γ_0. $C_n = 0$ is an equation of the x-axis for $n = 1, 3, 5, \cdots$, and $C_n = 0$ is an equation of the line $x = 2ab/(b-a)$ for $n = 2, 4, 6, \cdots$. This line passes through the point of coordinates

$$\left(\frac{2ab}{b-a}, 0\right),$$

which is denoted by E. Notice that if a circle touches α and β at two points P and Q, respectively, then the line PQ passes through E.

For the details, look the original papers with beautiful figures.

In the equation $g_z(x, y) = 0$, when we apply the division by zero calculus at $z = d$ we obtain the equation of ε. Meanwhile, in the equation $(z-d)g_z(x, y) = 0$, when we apply the division by zero calculus at $z = d$ we obtain the equation of γ_d.

Meanwhile, in the equation $g_z(x, y) = 0$ by letting $z \to \pm\infty$, we obtain the point of C.

The equation $g_z(x, y) = 0$ may be understood as an analytic motion of the circles γ_z with parameter z. Then, the problem may be considered as a general concept in mathematics.

As a simple and typical case, we will recall that for a general ordinary differential equation, we have a general solution with an arbitrary constant C; that is the general solution may be, in general, represented by using an analytic parameter.

For example, recall Clairau's differential equation

$$y = y'x + \frac{1}{y\prime}$$

and its general solutions containing any real number ξ are

$$y = x\xi + \frac{1}{\xi}.$$

Here note that as the singular solution of the differential equation and as the envelop of the general solutions, we have the parabolic curve

$$y^2 = 4x.$$

Firstly, note that for $\xi = 0$, we have $y = 0$ by the division by zero and it is a very natural solution of the Clairau's equation.

In connection with the Okumura's Laurent expansion, we have already the expansion at $\xi = 0$. Then, we obtain the results that

$$C_{-1} = 1, C_0 = -y, C_1 = x$$

and other Laurent expansion coefficients are all zero.

We see certainly that the coefficients C_0 and C_1 have their meanings for the general solutions. However, the Clairau's equation is given by the very general way

$$y = xy\prime + f(y'),$$

and so it certainly seems that the Okumura's Laurent expansion is mysterious.

When we consider the Laurent expansion of the function $g_z(x, y)$ we consider, of course, that x, y are fixed. However, when we consider the expansion with the equation $g_z(x, y) = 0$, x, y are depending on z.

Why the Okumura's Laurent expansions are so beautiful?

11 INTRODUCTION OF FORMULAS $\log 0 = \log \infty = 0$

For any fixed complex number a, we will consider the sector domain $\Delta_a(\alpha, \beta)$ defined by

$$0 \leq \alpha < \arg(z - a) < \beta < 2\pi$$

on the complex z plane and we consider the conformal mapping of $\Delta_a(\alpha, \beta)$ into the complex W plane by the mapping

$$W = \log(z - a).$$

Then, the image domain is represented by

$$S(\alpha, \beta) = \{W; \alpha < \Im W < \beta\}.$$

Two lines $\{W; \Im W = \alpha\}$ and $\{W; \Im W = \beta\}$ usually were considered as having the common point at infinity, however, in the division by zero, the point is represented by zero.

Therefore, $\log 0$ and $\log \infty$ **should be defined as zero**. Here, $\log \infty$ is precisely given in the sense of $[\log z]_{z=\infty}$. However, the properties of the logarithmic function should not be expected more, we should consider the value only. For example,

$$\log 0 = \log(2 \cdot 0) = \log 2 + \log 0$$

is not valid.

In particular, in many formulas in physics, in some expression, for some constants A, B

$$\log \frac{A}{B}, \tag{11.1}$$

if we consider the case that A or B is zero, then we should consider it in the form

$$\log \frac{A}{B} = \log A - \log B, \tag{11.2}$$

and we should put zero in A or B. Then, in many formulas, we will be able to consider the case that A or B is zero. For the case that A or B is zero, the identity (11.1) is not valid, then the expression $\log A - \log B$ may be valid in many physical formulas. However, the results are case by case, and **we should check the obtained results for applying the formula** (11.2) **for** $A = 0$ **or** $B = 0$**.** Then, we will be able to enjoy the formula apart from any logical problems as in the applications of the division by zero and division by zero calculus.

This paragraph will be unclear and so, we will show a typical example.

Inverse document frequency

The inverse document frequency is a measure of how much information the word provides, i.e., if it's common or rare across all documents. It is the logarithmically scaled inverse fraction of the documents that contain the word (obtained by dividing the total number of documents by the number of documents containing the term, and then taking the logarithm of that quotient):

$$idf(t, D) = \log \frac{N}{|\{d \in D; t \in d\}|}$$

where N is the total number of documents in the corpus $N = |D|$ and $\{d \in D; t \in d\}$ is the number of documents where the term appears (i.e., $tf(t, d) \neq 0$). If the term is not in the corpus, this will lead to a division-by-zero. It is therefore common to adjust the denominator to $idf(t, D) = \log N$, that is just our statement. See for the details

> tf–idf - Wikipedia https://en.wikipedia.org/wiki/Tf–idf Traduzir esta página In information retrieval, tf–idf or TFIDF, short for term frequency–inverse document frequency, is a numerical statistic that is intended to reflect how important a ... Motivations · Definition · Justification of idf · Example of tf–idf.

From the theory of the Sato hyperfunction theory:

From the theory of the hyperfunction theory, we have

$$\lim_{N\to\infty}\frac{1}{2\pi i}\log(N+1-z)=0$$

in the natural sense ([48], page 25).

11.1 Applications of $\log 0 = 0$

We can apply the result $\log 0 = 0$ for many cases as in the following way.

For example, we will consider the differential equation

$$y = xy' - \log y'.$$

We have the general solution

$$y = Cx - \log C$$

and the singular solution

$$y = 1 + \log x.$$

For $C = 0$, we have $y = 0$, by the division by zero, that is a reasonable solution.

For the differential equation

$$y' = 1 + \frac{y}{x},$$

we have the general solution

$$y = x(\log x + C).$$

How will be at $x = 0$? From

$$y' = \log x + C + 1$$

and

$$y'(0) = C + 1,$$

we have, for $x = 0$

$$\frac{y}{x} = C$$

and so, we see that for $x = 0$, the differential equation is satisfied.

For the differential equation

$$y' + \frac{1}{x}y = y^2 \log x,$$

we have the general solution

$$xy\{C - (\log x)^2\} = 2.$$

Dividing by C and by setting $C = 0$, by the division by zero, we have also the solutions $x = 0$ and $y = 0$.

For example, we will consider the differential equation

$$xy' = xy^2 - a^2 x \log^{2k}(\beta x) + ak \log^{k-1}(\beta x).$$

For the solution $y = a \log^{2k}(\beta x)$([94], page 95, **5**), we can consider the solution $y = 0$ as $\beta = 0$.

In the famous function (Leminiscate)

$$x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}, \quad a > 0,$$

we have

$$x = a \log \left[\frac{a + \sqrt{a^2 - y^2}}{y} \exp\left(-\frac{1}{a}\sqrt{a^2 - y^2} \right) \right].$$

By the division by zero, at the point $y = 0$

$$\left[\frac{a + \sqrt{a^2 - y^2}}{y} \exp\left(-\frac{1}{a}\sqrt{a^2 - y^2} \right) \right] = 0.$$

Thus the curve passes also the origin $(0,0)$.

In the differential equation

$$x^2y''' + 4x^2y'' - 2xy' - 4y = \log x,$$

we have the general solution

$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + C_3x^2 - \frac{1}{4}\log x + \frac{1}{4},$$

satisfying that at the origin $x = 0$

$$y(0) = \frac{1}{4}, y'(0) = 0, y''(0) = 2C_3, y'''(0) = 0.$$

We can give the values C_1 and C_2. For the sake of the division by zero, we can, in general, consider differential equations even at analytic and isolated singular points.

From the identities

$$Y_0(z) = \frac{2}{\pi}\left(\log\frac{1}{2}z + \gamma\right)J_0(z) +$$

$$\frac{2}{\pi}\left(\frac{z^2/2}{1!} - \frac{3}{2}(z^2/4)^2 + ...\right)$$

$$= \frac{4}{\pi^2}\int_0^{\pi/2}\cos(z\cos\theta)(\gamma + \log(2z + \sin^2\theta))d\theta$$

([1], 9.1.13 and 9.1.19), we have

$$Y_0(0) = \frac{2}{\pi}\gamma.$$

For the formula

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad (x > a > 0)$$

and

$$\cosh^{-1} z = \log(2z) - \frac{1}{4z^2} + ... \quad |z| > 1,$$

we have, for $a = 0$,

$$\int \frac{dx}{x} = \log 2 + \log x.$$

However, here in

$$\log 2 + \log x - \log a,$$

we have to have $\log 0 = 0$.

We will give a physical sense of $\log 0 = 0$. We shall consider a uniform line density μ on the $z-$ axis, then the force field $\mathbf{F}$ and the potential ϕ are given, for $\mathbf{p} = x\mathbf{i} + y\mathbf{j}, p = |\mathbf{p}|$,

$$\mathbf{F} = -\frac{2\mu}{p^2}\mathbf{p}$$

and

$$\phi = -2\mu \log \frac{1}{p},$$

respectively. On the z- axis, we have, of course,

$$\mathbf{F} = \mathbf{0}, \phi = 0.$$

O. Ufuoma introduced the example at (2019.12.28.5:39):

We consider the equation

$$2^x = 0,$$

then, from $x \log 2 = \log 0 = 0$, we have $x = 0$. Meanwhile, if

$$2^{(x-a)} = 0,$$

we have $x = a$. However, if $a \neq 0$, and if

$$2^{(x-a)} = 2^x 2^{-a},$$

then we have $x = 0$, a contradiction. Therefore, the identity is not valid in this case.

Anyhow, $\log 0 = 0$ is defined by a special sense and so, the derived results should be checked, case by case.

A finite part of divergence integrals

For a finite part of divergence integrals,

$$\log_{\varepsilon\to 0}\log\varepsilon = 0$$

is very natural and convenient. See [49], pages161-162.

11.2 Robin constant and Green's functions

From the typical case, we will consider a fundamental application. Let $D(a,R)=\{|z|>R\}$ be the outer disc on the complex plane. Then, the Riemann mapping function that maps conformally onto the unit disc $\{|W|<1\}$ and the point at infinity to the origin is given by

$$W=\frac{R}{z-a}.$$

Therefore, the Green function $G(z,\infty)$ of $D(a,R)$ is given by

$$G(z,\infty)=-\log\left\{\frac{R}{|z-a|}\right\}.$$

Therefore, from the representation

$$G(z,\infty)=-\log R+\log|z|+\log\left(1-\frac{a}{|z|}\right),$$

we have the identity

$$G(\infty,\infty)=-\log R,$$

that is the Robin constant of $D(a,R)$. This formula is valid in the general situation, because the Robin constant is defined by

$$\lim_{z\to b}\{G(z,b)+\log|z-b|\},$$

for a general Green function with pole at b of some domain ([2]).

11.3 Division by zero calculus for harmonic functions

For a harmonic function $h(z, a)$ with an isolated singular point at $z = a$ around $z = a$, we consider the analytic function

$$f(z, a) = A \log(z - a) + \sum_{n=-\infty}^{\infty} C_n (z - a)^n; \quad 0 < r < |z - a| < R,$$

whose real part is $h(z, a)$ with constants A and C_n.

Then, we **define** the division by zero calculus for the function $h(z, a)$ at $z = a$ by

$$h(a, a) = \Re C_0.$$

For example, for the Neumann function on the disc $|z| < R$ with the pole at $z = a$

$$N(z, a) = \log \frac{R^3}{|z - a||R^2 - \overline{a}z|},$$

we have

$$N(a, a) = \log \frac{R^3}{|R^2 - |a|^2|}.$$

For the famous Robin constant, this value seems not to be considered.

11.4 $e^0 = 1, 0$

By the introduction of the value $\log 0 = 0$, as the inversion function $y = e^x$ of the logarithmic function, we will consider that $y = e^0 = 0$. Indeed, we will show that this definition is very natural.

We will consider the conformal mapping $W = e^z$ of the strip

$$S(-\pi i, \pi i) = \{z; -\pi < \Im z < \pi\}$$

onto the whole W plane cut by the negative real line $(-\infty, 0]$. Of course, the origin 0 corresponds to 1. Meanwhile, we see that the negative line $(-\infty, 0]$ corresponds to the negative real line $(-\infty, 0]$. In particular, on the real line $\lim_{x\to-\infty} e^x = 0$. In our new space idea from the division by zero, the point at infinity is represented by zero and therefore, we should define as

$$e^0 = 0.$$

For the fundamental exponential function $W = \exp z$, at the origin, we should consider 2 values. The value 1 is the natural value as a regular point of the analytic function, meanwhile the value 0 is given with a strong discontinuity; however, this value will appear in the universe in a natural way.

For the elementary functions $y = x^n, n = \pm 1, \pm 2, \cdots$, we have

$$y = e^{n \log x}.$$

Then, we wish to have

$$y(0) = e^{n \log 0} = e^0 = 0.$$

As a typical example, we will consider the simple differential equation

$$\frac{dx}{x} - \frac{2ydy}{1+y^2} = 0.$$

Then, by the usual method,

$$\log|x| - \log|1+y^2| = C;$$

that is,

$$\log\left|\frac{x}{1+y^2}\right| = \log e^C = \log K, K = e^C > 0$$

and

$$\frac{x}{1+y^2} = \pm K.$$

However, the constant K may be taken as zero, as we see directly $\log e^C = \log K = 0$.

In the differential equations

$$y' = -\lambda e^{\lambda x} y^2 + a e^{\mu x} y - a e^{(\mu-\lambda)x}$$

and

$$y' = -b e^{\mu x} y^2 + a\lambda e^{\lambda x} y - a^2 b e^{(\mu+2\lambda)x}$$

we have solutions

$$y = -e^{-\lambda x},$$

$$y = a e^{\lambda x},$$

respectively. For $\lambda = 0$, as $y = -1$, $y = a$ are solutions, respectively, however, the functions $y = 0$, $y = 0$ are not solutions, respectively. However, many and many cases, as the function $y = e^{0 \cdot x} = 0$, we see that the function is solutions of differential equations, when $y = e^{\lambda \cdot x}$ is the solutions. See [94] for many concrete examples.

Meanwhile, we will consider the Fourier integral

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-\alpha|t|} dt = \frac{2\alpha}{\alpha^2 + \omega^2}.$$

For the case $\alpha = 0$, if this formula valid, then we have to consider $e^0 = 0$.

Furthermore, by Poisson's formula, we have

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + (2\pi n)^2}.$$

If $e^0 = 0$, then the above identity is still valid, however, for $e^0 = 1$, the identity is not valid. We have many examples.

For the integral

$$\int_0^{\infty} \frac{x^3 \sin(ax)}{x^4 + 4} dx = \frac{\pi}{2} e^{-a} \cos a,$$

the formula is valid for $a = 0$.

For the integral

$$\int_0^\infty \frac{\xi \sin(x\xi)}{1 + a^2\xi} d\xi = \frac{\pi}{2a^2} e^{-(x/a)}, \quad x > 0,$$

the formula is valid for $x = 0$.

For the identity

$$x^p + y^p = z^p,$$

for $p = 0$, we would like to consider $e^0 = 0$ from $x^p = \exp(p \log x)$.

Here, in particular, consider the cases: $p = 1/2$ and $x = 0$. Then, we have the natural result

$$0^{1/2} = \sqrt{0} = 0.$$

11.5 $0^0 = 1, 0$

By the standard definition, we will consider

$$0^0 = \exp(0 \log 0) = \exp 0 = 1, 0.$$

The value 1 is famous which was derived by N. Abel, meanwhile, H. Michiwaki had directly derived it as 0 from the result of the division by zero. However, we now know that $0^0 = 1, 0$ is the natural result.

We will see its reality.

For $0^0 = 1$: In general, for $z \neq 0$, from $z^0 = e^{0 \log z}$, $z^0 = 1$, and so, we will consider that $0^0 = 1$ in a natural way.

For example, in the elementary expansion

$$(1 + z)^n = \sum_{k=0}^{n} {}_nC_k z^k$$

the formula $0^0 = 1$ will be convenient for $k = 0$ and $z = 0$.

In the fundamental definition

$$\exp z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

in order to have a sense of the expansion at $z = 0$ and $k = 0$, we have to accept the formula $0^0 = 1$.

In the differential formula

$$\frac{d^n}{dx^n} x^n = n x^{n-1},$$

in the case $n = 1$ and $x = 0$, the formula $0^0 = 1$ is convenient and natural.

In the Laurent expansion, if $0^0 = 1$, it may be written simply as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n,$$

for $f(a) = C_0$.

For k: $k^2 < 1$, we have the identities

$$\sum_{n=0}^{\infty} k^n \sin(n+1)\theta = \frac{\sin\theta}{1 - 2k\cos\theta + k^2}$$

and

$$\sum_{n=0}^{\infty} k^n \cos(n+1)\theta = \frac{\cos\theta - k}{1 - 2k\cos\theta + k^2}.$$

In those identities, for $k = 0$, we have to have $0^0 = 1$.

For $0^0 = 0$: For any positive integer n, since $z^n = 0$ for $z = 0$, we wish to consider that $0^0 = 0$ for $n = 0$.

For the expansion

$$\frac{t}{\exp t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n,$$

with the Bernoulli's constants B_n, the usual value of the function at $t = 0$ is 1 and this meets the value $0^0 = 1$. Meanwhile, by the division by zero, we have the value 0 by the method

$$\frac{t}{\exp t - 1}\Big|_{t=0} = \frac{0}{\exp 0 - 1} = \frac{0}{0} = 0$$

and this meets with $0^0 = 0$. Note that by the division by zero calculus, we have the value 0 (V. V. Puha: 2018.7.3.6:01).

Philip Lloyed's question (2019.1.18): *What is the value of the equation*

$$x^x = x$$

?

By the equation

$$x(x^{x-1} - 1) = 0,$$

we have $x = 0$ and $x = 1$ therefore, we have also $0^0 = 0$.

P. Lloyed discovered also the solution -1, as we see the result directly and interestingly.

Khandakar Kawkabum Munir Saad asked the question for the equation $2^x = 0$ in Quora: 2019.7.4.17:00. We can give the solution $x = 0$. Therefore, for the very interesting equation $x^2 = 2^x$ we have the trivial solutions 2 and 4 and furthermore, the solution 0.

11.6 $\cos 0 = 1, 0$

Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

we wish to consider also the value $\cos 0 = 0$.

The values $e^0 = 0$ and $\cos 0 = 0$ may be considered that the values at the point at infinity are reflected to the origin and other many functions will have the same property.

The short version of this section was given by [54] in the Proceedings of the International Conference.

11.7 Finite parts of Hadamard in singular integrals

Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Together with these models, a variety of methods and applications for these integral equations has been developed. See, for example, [17, 34, 58, 61].

For the numerical calculation of this finite part, see [83], and there, they gave an effective numerical formulas by using the DE (double exponential) formula. See also its references for various methods.

For singular integrals, we will consider their integrals as divergence, however, the Haramard finite part or Cauhy's principal values give finite values; that is, from divergence values we will consider finite values; for this interesting property, we will be able to give a natural interpretation by the division by zero calculus.

What are singular integrals? For the interrelationship between divergence integrals and finite values in singular integrals, we can obtain an essential answer by means of the division by zero calculus.

Let $F(x)$ be an integrable function on an interval (c,d). The functions $F(x)/(x-a)^n (n = 1, 2, 3..., c < a < d)$ are, in general, not integrable on (c,d). However, for any $\epsilon > 0$, of course, the functions

$$\left(\int_c^{a-\epsilon} + \int_{a+\epsilon}^d\right) \frac{F(x)}{(x-a)^n} dx$$

are integrable. For an integrable function $\varphi(x)$ on (a,d), we assume the Taylor expansion

$$F(x) = \sum_{k=0}^{n-1} \frac{F^{(k)}(a)}{k!}(x-a)^k + \varphi(x)(x-a)^n.$$

Then, we have

$$\int_{a+\epsilon}^d \frac{F(x)}{(x-a)^n} dx$$

$$= \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{\epsilon^{n-k-1}} - \frac{F^{(n-1)}(a)}{(n-1)!} \log \epsilon$$

$$+\{- \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{(d-a)^{n-k-1}}$$

$$+\frac{F^{(n-1)}(a)}{(n-1)!} \log(d-a) + \int_{a+\epsilon}^{d} \varphi(x)dx\}.$$

Then, the last term {....} is the finite part of Hadamard of the integral

$$\int_a^d \frac{F(x)}{(x-a)^n} dx$$

and is written by

$$\text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx;$$

that is, precisely

$$\text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx$$

$$:= \lim_{\epsilon \to +0} \{ \int_{a+\epsilon}^{d} \frac{F(x)}{(x-a)^n} dx$$

$$- \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{\epsilon^{n-k-1}} + \frac{F^{(n-1)}(a)}{(n-1)!} \log \epsilon \}. \tag{11.3}$$

We do not take the limiting $\epsilon \to +0$, but we put $\epsilon = 0$, in (11.3), then we obtain, by the division by zero calculus, the formula

$$\text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx = \int_a^d \frac{F(x)}{(x-a)^n} dx.$$

The division by zero will give the natural meaning (**definition**) for the above two integrals.

Of course,

$$\text{f. p.} \int_c^d \frac{F(x)}{(x-a)^n} dx := \text{f. p.} \int_c^a \frac{F(x)}{(x-a)^n} dx$$

$$+ \quad \text{f. p.} \int_a^d \frac{F(x)}{(x-a)^n} dx.$$

When $n = 1$, the integral is the Cauchy principal value.

In particular, for the expression (11.3), we have, missing $\log \epsilon$ term, for $n \geq 2$

$$\text{f. p.} \int_c^d \frac{F(x)}{(x-a)^n} dx$$

$$= \lim_{\epsilon \to +0} \{ (\int_c^{a-\epsilon} + \int_{a+\epsilon}^d) \frac{F(x)}{(x-a)^n} dx$$

$$- \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1+(-1)^{n-k}}{\epsilon^{n-k-1}} \}.$$

11.8 Complex function $\log z$

The function $f(z) = \log z$ is, of course, multiply-valued, however, we will define

$$f(0) = \log 0 = 0,$$

as a single-valued at $z = 0$.

Note, by the division by zero calculus,

$$\frac{z^n}{n}|_{n=0} = \log z.$$

Then, for

$$g(z) = \frac{1}{z}$$

and

$$G(z) = \log z + C,$$

we have

$$g(0) = 0$$

and

$$G(0) = C.$$

Note that the identity

$$\arg \overline{z} = -\arg z,$$

and so, if the function $\arg z$ is extensible to the origin as an odd function, then the value $\arg 0$ has to be zero and note that

$$\log z = \log |z| + i \arg z.$$

11.9 Complex function $\arg z$

We will show the examples of $\arg z = 0$; however, this should be considered as a convention as in $\log 0 = 0$.

On the complex z plane we consider the unit circle with its center $z = 0$. On the unit circle we consider the function

$$\arg \frac{z+i}{z-i}.$$

Of course, it is $\pi/2$ except two points $z = i, -i$. However, for those points, the values should be still $\pi/2$. The function should be considered for the points $z = i, -i$.

For example, for the point $z = i$,

$$\arg \frac{z+i}{z-i}|_{z=i} = \arg 2i - \arg 0 = \frac{\pi}{2}.$$

We should consider as in the above and not apply the division by zero calculus as in

$$\arg \frac{z+i}{z-i}|_{z=i} = \arg 1.$$

We consider the bounded harmonic function

$$v(z) = \pi + 2\arg(1 - z)$$

on the unit disc $|z| < 1$ having the boundary values θ at the boundary points $e^{i\theta}$ $(0 < \theta < 2\pi)$. Then we have

$$v(1) = \pi.$$

This result is a very reasonable, because 1 is $e^{oi} = e^{2\pi i}$ and π is the mean value of 0 and 2π.

Meanwhile, we will consider the harmonic measure of the unit disc taking the boundary value 1 on the arc $\theta_0 < \theta < \theta_1$ and the value 0 on the arc $\theta_1 < \theta < \theta_0$

$$\omega(z) = \frac{1}{\pi}\arg\frac{e^{i\theta_1} - z}{e^{i\theta_0} - z} - \frac{\theta_1 - \theta_0}{2\pi}.$$

Since the radial limits at the points $e^{i\theta_0}$ and $e^{i\theta_1}$ are 1/2, it seems that the division by zero and division by zero calculus do not have meanings of the function

$$\frac{e^{i\theta_1} - z}{e^{i\theta_0} - z}$$

at the points $e^{i\theta_0}$ and $e^{i\theta_1}$.

12 Divergence series and integrals from the viewpoint of the division by zero calculus

For the fundamental expansion on $|z| < 1$

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \tag{12.1}$$

we will state a new interpretation.Of course, we know its meaning of the expansion (12.1) that is valid in the open unit disc $|z| < 1$. Now, how will be the expansion for the point at $z = 1$? Usually, we will consider that

$$\lim_{z \to 1-0} \frac{1}{1-z}$$

diverges to infinity, meanwhile,

$$\lim_{N \to \infty} \sum_{j=0}^{N} 1^j = +\infty.$$

Of course, these are right. However, now we can consider that by the division by zero calculus

$$\frac{1}{1-z} = 0$$

at the point $z = 1$ and

$$\lim_{N \to \infty} \sum_{j=0}^{N} 1^j = 0.$$

Then the expansion (12.1) is still valid for the point $z = 1$. For the limiting

$$\lim_{N \to \infty} \sum_{j=0}^{N} 1^j$$

we have two meanings, as we see already.

Next, for the integral

$$\int_1^\infty \frac{1}{x} dx, \tag{12.2}$$

we will consider that

$$\int_1^\infty \frac{1}{x} dx = \lim_{R\to\infty} \int_1^R \frac{1}{x} dx = \lim_{R\to\infty} \log R = +\infty. \tag{12.3}$$

However, by the new idea of the point at infinity, this is zero ([54]), since $\log \infty = 0$.

For

$$\int_0^\infty J_0(\lambda t) dt = \frac{1}{\lambda}, \tag{12.4}$$

we have, by the division by zero calculus idea,

$$\int_0^\infty J_0(0t) dt = \int_0^\infty 1 dt = \frac{1}{\lambda}|_{\lambda=0} = 0. \tag{12.5}$$

We can find many and many formulas in the divergence series and integrals that may be applied the idea. We will show examples.

In the formula ([21], page 153), for $0 \le x, t \le \pi$

$$\sum_{n=1}^\infty \frac{\sin ns \sin nt}{n} = \frac{1}{2} \log |\sin((s+t)/2)/\sin((s-t)/2)|,$$

for $s = t = 0, \pi$, we can obtain that

$$0 = \frac{1}{2} \log \frac{0}{0} = \log 0.$$

In general, for $s = t$, we may consider that

$$\sum_{n=1}^\infty \frac{\sin^2 ns}{n} = \frac{1}{2} \log |\sin((s+s)/2)/0)|$$

$$= \frac{1}{2} \log |\sin ns/0| = \frac{1}{2} \log 0 = 0.$$

By Poisson's formula, we have

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + (2\pi n)^2}.$$

For $\alpha = 0$, the both sides are zero.

For the integrals, for non-negative integer n and $a > 0$,

$$\int_0^{\infty} t^{2n} e^{-at^2} dt = \frac{\Gamma(n + (1/2))}{2a^{n+1}}$$

and

$$\int_0^{\infty} t^{2n+1} e^{-at^2} dt = \frac{n!}{2a^{n+1}},$$

for $a = 0$, we see that they are zero.

Meanwhile, from the well-known expansion ([1], page 807) of the Riemann zeta function

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2(s-1)^2 + ...,$$

we see that the value at $s = 1$ is the Euler constant γ; that is,

$$\zeta(1) = \gamma. \tag{12.6}$$

Meanwhile, from the expansion

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} C_k \frac{z^{2k-1}}{2k-1}$$

([1], 635 page 18.5.5), we have

$$\zeta(0) = 0.$$

From our idea that the point at infinity is represented by zero, the result $\zeta(1) = \gamma$ is contrary curious. However, we should consider that

$$\sum_{n=1}^{\infty} \frac{1}{n} = 0, \tag{12.7}$$

because the limit tends to infinity or N terms sums are not bounded. We should consider some problem for its representation $\zeta(1)$ by the Riemann zeta function by its analytic extension. For this point, recall that

$$\zeta(z) = -\frac{1}{12} + (z+1)\left(\frac{1}{12} - \log C\right) + ...$$

for $z = -1$; that is $\zeta(-1) = -1/12$ by the division by zero calculus, of course, it does not represent the series

$$1 + 2 + 3 + ... + n + ...$$

In many places in mathematical science, infinity notation may be replaced with zero, by the concept of the division by zero calculus. How to represent the formulas? We will need some time in order to see some global situation. These facts anyhow will show a new meaning of ZERO.

13 BASIC MEANINGS OF VALUES AT ISOLATED SINGULAR POINTS OF ANALYTIC FUNCTIONS

The values of analytic functions at isolated singular points were given by the coefficients C_0 of the Laurent expansions (the first coefficients of the regular part) as the division by zero calculus. Therefore, their property may be considered as arbitrary ones by any sift of the image complex plane. Therefore, we can consider the values as zero in any Laurent expansions by shifts, as normalizations. However, if by another normalizations, the functions (the Laurent expansions) are determined, then the values C_0 will have their senses. We will firstly examine such properties for the Riemann mapping function.

Let D be a simply-connected domain containing the point at infinity having at least two boundary points. Then, by the celebrated theorem of Riemann, there exists a uniquely determined conformal mapping with a series expansion

$$W = f(z) = C_1 z + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots, \quad C_1 > 0, \quad (13.1)$$

around the point at infinity which maps the domain D onto the exterior $|W| > 1$ of the unit disc on the complex W plane. We can normalize (13.1) as follows:

$$\frac{f(z)}{C_1} = z + \frac{C_0}{C_1} + \frac{C_{-1}}{C_1 z} + \frac{C_{-2}}{C_1 z^2} + \ldots.$$

Then, this function $\frac{f(z)}{C_1}$ maps D onto the exterior of the circle of radius $1/C_1$ and so, it is called the **mapping radius** of D. See [8, 122]. Meanwhile, from the normalization

$$f(z) - C_0 = C_1 z + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots,$$

by the natural shift C_0 of the image plane, the unit circle is mapped to the unit circle with its center C_0. Therefore, C_0 may be called as **mapping center** of D. The function $f(z)$ takes the value C_0 at the point at infinity in the sense of the division by zero calculus and now we have its natural sense by the mapping center of D. We have considered the value of the function $f(z)$ as infinity at the point at infinity, however, practically it was the value C_0. This will mean that in a sense the value C_0 is the farthest point from the point at infinity or the image domain with the strong discontinuity.

The properties of mapping radius were investigated deeply in conformal mapping theory like estimations, extremal properties and meanings of the values, however, it seems that there is no information on the property of mapping center. See many books on conformal mapping theory or analytic function theory. See [122] for example.

From the fundamental Bierberbach area theorem, we can obtain the following inequality:

For analytic functions on $|z| > 1$ with the normalized expansion around the point at infinity

$$g(z) = z + b_0 + \frac{b_1}{z} + \cdots$$

that are univalent and take no zero point,

$$|b_0| \leq 2.$$

In our sense

$$g(\infty) = b_0.$$

See [63], Chapter V, Section 8 for the details.

13.1 Values of typical Laurent expansions

The values at singular points of analytic functions are represented by the Cauchy integral, and so for given functions, the calculations will be simple numerically, however, their analytical (precise) values will be given by using the known Taylor or

Laurent expansions. In order to obtain some feelings for the values at singular points of analytic functions, we will see typical examples and fundamental properties.

For
$$f(z) = \frac{1}{\cos z - 1}, \quad f(0) = -\frac{1}{6}.$$
Here, note that
$$\frac{1}{\cos z - 1} = -\frac{1}{z^2} - \frac{1}{6} - \frac{z^2}{120} - \cdots.$$

For
$$f(z) = \frac{\log(1+z)}{z^2}, \quad f(0) = \frac{-1}{2}.$$

For
$$f(z) = \frac{1}{z(z+1)}, \quad f(0) = -1.$$

For our purpose in the division by zero calculus, when a is an isolated singular point, we have to consider the Laurent expansion on $\{0 < r < |z-a| < R\}$ such that r may be taken arbitrary small r, because we are considering the function at a.

For
$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}, \quad f(i) = \frac{1}{4}.$$

For
$$f(z) = \frac{1}{\sqrt{(z+1)} - 1}, \quad f(0) = \frac{1}{2}.$$

For the Bernoulli constants B_n, we have the expansions
$$\frac{1}{(\exp z) - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{(2n)!} z^{2n-1}$$
$$= \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4\pi^2 n^2}$$

and so, we obtain

$$\frac{1}{(\exp z) - 1}(z = 0) = -\frac{1}{2},$$

([96], page 444).

From the well-known expansion ([1], page 807) of the Riemann zeta function

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2(s-1)^2 + ...,$$

we see that the Euler constant γ is the value at $s = 1$; that is,

$$\zeta(1) = \gamma.$$

Meanwhile, from the expansion

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} C_k \frac{z^{2k-1}}{2k-1}$$

([1], 635 page 18.5.5), we have

$$\zeta(0) = 0.$$

From the representation of the Gamma function $\Gamma(z)$

$$\Gamma(z) = \int_1^{\infty} e^{-t} t^{z-1} dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}$$

([96], page 472), we have

$$\Gamma(-m) = E_{m+1}(1) + \sum_{n=0, n \neq m}^{\infty} \frac{(-1)^n}{n!(-m+n)}$$

and

$$[\Gamma(z) \cdot (z+n)](-n) = \frac{(-1)^n}{n!}.$$

In particular, we obtain

$$\Gamma(0) = -\gamma,$$

by using the identity

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{nn!}, \quad |\arg z| < \pi$$

([1], 229 page, (5.1.11)). Of course,

$$E_1(z) = \int_z^{\infty} e^{-t} t^{-1} dt.$$

From the recurrence formula

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

of the Psi (Digamma) function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

([1], 258), we have, for $z = 0, 1$,

$$\psi(0) = \psi(1) = -\gamma.$$

Note that

$$\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}, \quad |z| < 1$$

$$= -\gamma + \sum_{n=}^{\infty} \frac{z}{n(n+z)}, \quad z \neq -1, -2, ...$$

([1], 259).

From the identity

$$\frac{1}{\psi(z+1) - \psi(z)} = z,$$

we have

$$\frac{1}{\psi(z+1)-\psi(z)}(z=0)=0.$$

From the identities

$$\frac{\Gamma(z)}{\Gamma(z+1)}=\frac{1}{z},$$

and

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin \pi z},$$

note that their values are zero at $z=0$.

By using WolframAlpha, we obtain:

$$\Gamma(x)=\frac{1}{x}-\gamma+\frac{1}{12}(6\gamma^2+\pi^2))x+...,$$

$$\Gamma(x)=\frac{1}{x+1}+(\gamma-1)+\left(-1+\gamma-\frac{1}{2}\gamma^2-\frac{1}{12}\pi^2\right)(x+1)+...,$$

$$\Gamma(x)=\frac{1}{2(x+2)}+\frac{1}{4}(3-2\gamma)+\frac{1}{24}\left(21-18\gamma+6\gamma^2+\pi^2\right)(x+2)+...,$$

$$\Gamma(x)=-\frac{1}{6(x+3)}+\frac{1}{36}(6\gamma-11)$$

$$+\frac{1}{216}\left(-85+66\gamma-18\gamma^2-3\pi^2\right)(x+3)+...,$$

$$\psi(z)=-\frac{1}{z}-\gamma+\frac{1}{6}\pi^2 z+...,$$

$$\psi(z)=-\frac{1}{z+1}+(1-\gamma)+\left(1+\frac{1}{6}\pi^2\right)(z+1)+...,$$

$$\psi(z)=-\frac{1}{z+2}+\left(\frac{3}{2}-\gamma\right)+\left(\frac{5}{4}+\frac{1}{6}\pi^2\right)(z+2)+...,$$

and

$$\psi(z)=-\frac{1}{z+3}+\left(\frac{11}{6}-\gamma\right)+\left(\frac{49}{36}+\frac{1}{6}\pi^2\right)(z+3)+....$$

From the expansions

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} C_k z^{2k-2}$$

and

$$\wp'(z) = \frac{-2}{z^3} + \sum_{k=2}^{\infty} (2k-2) C_k z^{2k-3}$$

([1], 623 page, 18.5.1. and 18.5.4), we have

$$\wp(0) = \wp'(0) = 0.$$

We can consider many special functions and the values at singular points. For example,

$$Y_{3/2}(z) = J_{-3/2}(z) = -\sqrt{\frac{2}{\pi z}} \left(\sin z + \frac{\cos z}{z} \right),$$

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z,$$

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

and so on. They take the value zero at the origin, however, we can consider some meanings of the value.

Of course, the product property is, in general, not valid:

$$f(0) \cdot g(0) \neq (f(z)g(z))(0);$$

indeed, for the functions $f(z) = z + 1/z$ and $g(z) = 1/z + 1/(z^2)$

$$f(0) = 0, g(0) = 0, (f(z)g(z))(0) = 1.$$

For an analytic function $f(z)$ with a zero point a, for the inversion function

$$(f(z))^{-1} := \frac{1}{f(z)},$$

we can calculate the value $(f(a))^{-1}$ at the singular point a.

For example, note that for the function

$$f(z) = z - \frac{1}{z},$$

$f(0) = 0, f(1) = 0$ and $f(-1) = 0$. Then, we have

$$(f(z))^{-1} = \frac{1}{2(z+1)} + \frac{1}{2(z-1)}.$$

Hence,

$$((f(z))^{-1})(z = 0) = 0, ((f(z))^{-1})(z = 1) = \frac{1}{4},$$

$$((f(z))^{-1}(z = -1) = -\frac{1}{4}.$$

Here, note that the point $z = 0$ is not a regular point of the function $f(z)$.

We, meanwhile, obtain that

$$\left(\frac{1}{\log x}\right)_{x=1} = 0.$$

Indeed, we consider the function $y = \exp(1/x), x \in \mathbf{R}$ and its inverse function $y = \frac{1}{\log x}$. By the symmetric property of the functions with respect to the function $y = x$, we have the desired result.

Here, note that for the function $\frac{1}{\log x}$, we can not use the Laurent expansion around $x = 1$, and therefore, the result is not trivial.

Meanwhile, by the same idea, we will have the results

$$\left(\frac{1}{\log x}\right)_{x=0} = 0.$$

and

$$\exp \frac{1}{z}(0) = 0.$$

The function $\exp \frac{1}{z}$ is a two-valued function at $z = 0$ of 1 and 0.

Justin Lee asked the value of the function

$$y = x^{1/\log x}$$

at $x = 0$ on 2020.4.14 on You Tube. We answered that the values are 0 and 1, two valued.

We shall refer to the trigonometric functions. See, for example, ([1], page 75) for the expansions.

From the expansions

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty} (-1)^{\nu} \left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi} \right)$$

and

$$= \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots,$$

$$\left(\frac{1}{\sin z} \right)(0) = 0.$$

Meanwhile, from the expansions

$$\frac{1}{\sin^2 z} = \sum_{\nu=-\infty}^{\infty} \frac{1}{(z - \nu\pi)^2}$$

and

$$= \frac{1}{z^2} + \frac{1}{3} + \frac{z^2}{15} + \cdots,$$

$$\left(\frac{1}{\sin^2 z} \right)(0) = \frac{2}{\pi^2} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{1}{3}.$$

From the expansion

$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty} (-1)^{\nu} \left(\frac{1}{z - (2\nu - 1)\pi/2} + \frac{2}{(2\nu - 1)\pi} \right)$$

$$= -\frac{1}{z-\frac{\pi}{2}} - \frac{1}{6}\left(z-\frac{\pi}{2}\right) - \frac{7}{360}\left(z-\frac{\pi}{2}\right)^3 + \cdots,$$

$$\left(\frac{1}{\cos z}\right)\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1} = 0.$$

Meanwhile, from the expansion

$$\frac{1}{\cos^2 z} = \sum_{\nu=-\infty}^{+\infty}\frac{1}{(z-(2\nu-1)\pi/2)^2}$$

and

$$= \frac{1}{\left(z-\frac{\pi}{2}\right)^2} + \frac{1}{3} + \frac{1}{15}\left(z-\frac{\pi}{2}\right)^2 + \cdots,$$

$$\left(\frac{1}{\cos^2 z}\right)\left(\frac{\pi}{2}\right) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

By the Laurent expansion and by the definition of the division by zero calculus, we note that:

Theorem: *For any analytic function $f(z)$ on $0 < |z| < \infty$, we have*

$$f(0) = f(\infty).$$

For a rational function

$$f(z) = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}; \quad a_0, b_0 \neq 0; \quad a_m, b_n \neq 0, m, n \geq 1$$

$$f(0) = f(\infty) = \frac{a_0}{b_0}.$$

Of course, here $f(\infty)$ is not given by any limiting $z \to \infty$, but it is the value at the point at ∞.

The derivatives of $n!$:

Note that the identity $z! = \Gamma(z+1)$ and the Gamma function is a meromorphic function with isolated singular points on the entire complex plane. Therefore, we can consider the derivatives of the Gamma function even at isolated singular points, in our sense.

13.2 Values of domain functions

In this subsection, we will examine the values of typical domain functions at singular points. For a basic reference, see [63].

1). For the mapping

$$W = \frac{z}{1-z}$$

that maps conformally the unit disc $|z| < 1$ onto the half-plane $\{ReW > \frac{1}{2}\}$, we have

$$W(1) = -1.$$

2). For the Koebe function

$$W = \frac{z}{(1-z)^2}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of $(-\infty, -\frac{1}{4})$ we have

$$W(1) = 0.$$

We can understand it as follows. The boundary point $z = 1$ of the unit disc is mapped to the point at infinity, however, the point is represented by zero. We can see the similar property, for many cases.

3). For the Joukowsky transform

$$W = \frac{1}{2}\left(\frac{1}{z} + z\right)$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of $[-1, 1]$ we have

$$W(0) = 0.$$

This correspondence will be curious in a sense. The origin that is an interior point corresponds to the boundary point of the origin. Should we consider the situation as in the case 2? - the image of the origin is the point at infinity and the point is represented by zero, the origin.

4). For the transform

$$W = \frac{z}{1 - z^2}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of the imaginary axis of $[+\infty, i/2]$ and $[-\infty, -i/2]$ we have

$$W(1) = -\frac{1}{4}, \quad W(-1) = \frac{1}{4},$$

by the method of Laurent expansion method, curiously. Should we consider the values at $z = 1$ and $z = -1$ as 0 from $1/0$ and $-1/0$ by the insertings $z = 1$ and $z = -1$ in the numerator and denominator?

5). For the conformal mapping $W = P(z; 0, v), |v| < 1$ of the unit disc onto the circular slit W plane that is normalized by $P(0; 0, v) = 0$ and

$$P(z : 0, v) = \frac{1}{z - v} + C_0 + C_0(z - v) + ...,$$

is given by, explicitly

$$P(z; 0, v) = \frac{1}{v(1 - |v|^2)} \frac{z(1 - \overline{v}z)}{z - v}$$

([63], 340 page). Then, we obtain

$$P(z : 0, v)|_{z=v} = C_0 = \frac{1 - 2|v|^2}{v(1 - |v|^2)},$$

at $z = v$ by the Laurent expansion method. By the constant C_0, we can consider as in the mapping center by shift of the image plane. We may also give the value for $z = v$ by

$$P(z : 0, v)|_{z=v} = \left[\frac{1}{v(1-|v|^2)} \frac{z(1-\overline{v}z)}{z-v}\right]_{z=v} = \frac{v(1-|v|^2)}{0} = 0.$$

The circumstance is similar for the corresponding canonical conformal mapping $Q(z : 0, v)$ for the radial slit mapping.

The Szegö kernel

For the Szegö kernel $K(z, \overline{u})$ and its adjoint L kernel $L(z, u)$ on a regular region D on the complex z plane, the function

$$f(z) = \frac{K(z, \overline{u})}{L(z, u)}$$

is the Ahlfors function on the domain D and it maps the domain D onto the unit disc $|w| < 1$ with one to the multiplicity of the connectivity of the domain D. From the relation $L(z, u) = -L(u, z)$, we see that $L(u, u) = 0$ in the sense of the division by zero calculus. Therefore, from the identity

$$L(z, u) = \frac{1}{2\pi(z-u)} + \frac{1}{2\pi} \int_{\partial D} \frac{K(u, \overline{\zeta})}{\zeta - z} |d\zeta|$$

([63], 390 page), we have the identity

$$\int_{\partial D} \frac{K(z, \overline{\zeta})}{\zeta - z} |d\zeta| = 0.$$

These results can also be confirmed directly.

By this method, we can find many new identities.

13.3 The values of the Riemann zeta function at positive integers

In this subsection, we will examine the values of the Riemann zeta function for positive integers $2 \leq n$ by using the division by zero calculus. For the values of the Riemann zeta functions at positive integers, see [31]. In particular, note that for odd integers, their values were mysterious. We can give the values in the both senses of analytical and numerical.

13.3.1 Simple applications of the division by zero calculus

As the first try, we will see some simple applications of the division by zero calculus to some typical formulas in order to look for the values of the Riemann zeta function.

Method 1

First, we recall the basic identity

$$\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k\pi)^2}$$

([1], page 75: 4.3.92), because the right hand side becomes, by the division by zero calculus, at $z = 0$, by taking n times derivative

$$\frac{2(n-1)!}{\pi^n}\zeta(n).$$

However, note that this formula is valid for an even n.

Meanwhile, we will use the expansion

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots$$

$$+\frac{(-1)^{n-1}2(2^{2n-1}-1)B_{2n}}{(2n)!}z^{2n-1} + \cdots \quad (|z| < \pi) \tag{13.2}$$

([1], page 75: 4.3.68). We will calculate the square of this expansion and by taking n order derivative, we can calculate the value at $z = 0$, by the division by zero calculas.

We can obtain simply the following results, by this method

$$\zeta(2) = \frac{\pi^2}{6},$$

$$\eta(2) = \frac{\pi^2}{12}$$

and

$$\zeta(4) = \frac{\pi^4}{90}.$$

For the values of the Riemann zeta function for even integers, we know good results, and so we do not examine any further details here.

Method 2

We will use the identities:

$$\cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2\pi^2} \tag{13.3}$$

([1], page 75, 4.3.93) and

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots \tag{13.4}$$

$$-\frac{(-1)^{n-1}2^{2n}B_{2n}}{(2n)!}z^{2n-1} - \cdots \quad (|z| < 1)$$

([1], page 75, 4.3.70).

From (13.3), we have

$$\left(\frac{\cot z}{z}\right)' = -\frac{2}{z^3} - 4z \sum_{k=1}^{\infty} \frac{1}{(z^2 - k^2\pi^2)^2}.$$

Therefore, we have

$$\left(\frac{\cot z}{z}\right)' \frac{1}{z} = -\frac{2}{z^4} - 4 \sum_{k=1}^{\infty} \frac{1}{(z^2 - k^2\pi^2)^2}$$

$$= -\frac{2}{z^4} - \frac{2}{45} -$$

Hence, we have

$$\zeta(4) = \frac{\pi^4}{90}.$$

By induction, we can obtain $\zeta(2n)$.

Method 3

We will use the identities:

$$\frac{1}{\sin z} = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2\pi^2} \tag{13.5}$$

([1], page 75, 4.3.93) and (13.2).

From these identities, we obtain

$$\frac{1}{z^2} + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2\pi^2}$$

$$= \frac{1}{z^2} + \frac{1}{6} + \frac{7}{360} + \cdots$$

Therefore, we obtain the result

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{5^2} + \cdots = \frac{\pi^2}{12}.$$

Method 4

Recall the expansion

$$\cot z = \frac{1}{z} + \sum_{k=-\infty, k\neq 0}^{\infty} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi} \right). \tag{13.6}$$

By taking n order derivatives that are very simple with (13.3) we obtain the values of the Riemann zeta function $\zeta(n)$, easily. However, note that n has to be even integers.

13.3.2 Some general definite result

Recall the expansion

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{k=1}^{\infty} \left(\frac{1}{z+k} - \frac{1}{k} \right) \tag{13.7}$$

([19], page 53). We obtain, taking $n-1; (n > 2)$ order derivative, by the division by zero calculus

$$\zeta(n) = \frac{(-1)^n}{(n-1)!} \psi^{(n-1)}(z)|_{z=0}. \tag{13.8}$$

Recall the expansion

$$\psi(z+1) = -\gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k) z^{k-1} \quad (|z| < 1) \tag{13.9}$$

([1], page 259, 6.3.14). Then we obtain

$$\frac{\psi(z+1)}{z^{n-1}} = \frac{-\gamma}{z^{n-1}} + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^{k-1}}{z^{n-1}} \quad (|z| < 1). \tag{13.10}$$

Hence, by the division by zero calculus, we obtain, for $n > 2$

$$\zeta(n) = (-1)^n \frac{\psi(z+1)}{z^{n-1}}|_{z=0}. \tag{13.11}$$

Then, by using (13.9), we obtain for $n = 2$, by MATHEMATICA

$$\zeta(3) = 1 - \frac{\psi^{(2)}(2)}{2} \sim 1.20206.$$

Note that with MATHEMATICA, we can derive the Laurent expansion for many analytic functions and so we can obtain the division by zero calculus for many analytic functions.

In general, we have

Theorem:

$$\zeta(n) = 1 - \frac{\psi^{(n-1)}(2)}{(n-1)!}. \tag{13.12}$$

These values may be calculated easily as follows:

$$\zeta(5) = 1 - \frac{1}{24}\psi^{(4)}(2) \sim 1.03693,$$

$$\zeta(6) = \frac{\pi^6}{945} \sim 1.01734$$

$$\zeta(7) = 1 - \frac{1}{720}\psi^{(6)}(2) \sim 1.00835,$$

$$\zeta(8) = \frac{\pi^8}{9450} \sim 1.00408,$$

$$\zeta(9) = 1 - \frac{1}{40320}\psi^{(8)}(2) \sim 1.00201.$$

Note that the value of the function $\psi(z)$ may be calculated easily by MATHEMATICA.

In particular, note the well-known formulas:

$$\zeta(2n) = (-1)^{n+1}\,\frac{B_{2n}(2\pi)^{2n}}{2(2n)!},$$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}.$$

The following formula was given by S. Ramanujan (1887 - 1920)

$$\zeta(2n+1) = 2^{2n}\,\pi^{2n+1}\sum_{k=0}^{n+1}(-1)^{k+1}\,\frac{B_{2k}}{(2k)!}\,\frac{B_{2n+2-2k}}{(2n+2-2k)!}$$

$$-2\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\pi k}-1}.$$

For the known results, our results are the similar.

This subsection is taken from ([89]).

13.4 Mysterious properties on the point at infinity

In this subsection, we will refer to some feelings on the point at infinity, because, the division by zero creates a new world on the point at infinity.

13.4.1 Many points at infinity?

When we consider a circle with its center P, by the inversion with respect to the circle, the points of a neighborhood at the point P are mapped to a neighborhood around the point at infinity except the point P. This property is independent of the radius of the circle. It looks that the point at infinity is depending on the center P. This will mean that there exist many points at infinity, in a sense.

13.4.2 Stereographic projection

The point at infinity may be realized by the stereographic projection as well known. However, the projection is depending on the position of the sphere (the plane coordinates). Does this mean that there exist many points at infinity?

13.4.3 Laurent expansion

From the definition of the division by zero calculus, we see that if there exists a negative n term in (5.5)

$$\lim_{z\to a} f(z) = \infty,$$

however, we have (5.6). The values at the point a have many values, that are any complex numbers. At least, **in this sense**, we see that we have many points at the point at infinity.

In the sequel, we will show typical points at infinity.

13.4.4 Diocles' curve of Carystus (BC 240? - BC 180?)

The beautiful curve

$$y^2 = \frac{x^3}{2a - x}, \quad a > 0$$

is considered by Diocles. By setting $X = \sqrt{2a - x}$ we have

$$y = \pm \frac{x^{(3/2)}}{\sqrt{2a - x}} = \pm \frac{(2a - X^2)^{(3/2)}}{X}.$$

Then, by the division by zero calculus at $X = 0$, we have a reasonable value 0.

Meanwhile, for the function $\frac{x^3}{2a-x}$, we have $-12a^2$, by the division by zero calculus at $x = 2a$. This leads to a wrong value.

13.4.5 Nicomedes' curve (BC 280 - BC 210)

The very interesting curve

$$r = a + \frac{b}{\cos\theta}$$

is considered by Nicomedes from the viewpoint of the 1/3 division of an angle. That has very interesting geometrical meanings. For the case $\theta = \pm(\pi/2)$, we have $r = a$, by the division by zero calculus.

Of course, the function is symmetric for $\theta = 0$, however, we have a mysterious value $r = a$, for $\theta = \pm(\pi/2)$. Look the beautiful graph of the function.

13.4.6 Newton's curve (1642 - 1727)

Meanwhile, for the famous Newton curve

$$y = ax^2 + bx + c + \frac{d}{x} \quad (a, d \neq 0),$$

of course, we have $y(0) = c$.

Meanwhile, in the division by zero calculus, the value is determined by the information around any analytical point for an analytic function, as we see from the basic property of analytic functions.

At this moment, the properties of the values of analytic functions at isolated singular points are mysterious, in particular, in the geometrical sense.

13.4.7 Unbounded, however, bounded

We will consider the high

$$y = \tan\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

on the line $x = 1$. Then, the high y is unbounded, however, the high line (gradient) can not be extended beyond the y axis. The restriction is given by $0 = \tan(\pi/2)$.

Recall the stereographic projection of the complex plane. The points on the plane can be expanded in an unbounded way, however, all points on the complex plane have to be corresponded to the points of the Riemann sphere. The restriction is the point at infinity which corresponds to the north pole of the Riemann sphere and the point at infinity is represented by 0.

This subsection is presented in [105].

13.5 Differential coeffcients and residures

We note the basic relation for analytic functions $f(z)$ for the analytic extension of $f(x)$ to complex variable z

$$\frac{f(x)}{(x-a)^n}|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res._{\zeta=a}\left\{\frac{f(\zeta)}{(\zeta-a)^{n+1}}\right\}.$$

We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residures. In this viewpoint, in particular, from the differential coefficient properties stated as in the above, we can derive the correspondent properties.

For example, for the product case, we have

$$Res._{\xi=x}\frac{(fg)(\xi)}{(\xi-x)^2}$$

$$= Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}g(x) + f(x)Res._{\xi=x}\frac{g(\xi)}{(\xi-x)^2}.$$

For the rational case

$$Res._{\xi=x}\frac{(f/g)(\xi)}{(\xi-x)^2}$$

$$= \frac{Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}g(x) - f(x)Res._{\xi=x}\frac{g(\xi)}{(\xi-x)^2}}{g(x)^2}.$$

For the inverse function $x = f^{-1}(y)$ for the function $y = f(x)$,

$$Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}Res._{\eta=y}\frac{f^{-1}(\eta)}{(\eta-y)^2} = 1.$$

For the second order derivatives, we have, for $x = g(t)$

$$Res._{\tau=t}\frac{f(g(\tau))}{(\tau-t)^3} = Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^3}Res._{\tau=t}\frac{g(\tau)}{(\tau-t)^2}$$

$$+Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}Res._{\tau=t}\frac{g(\tau)}{(\tau-t)^3}$$

and

$$Res._{\xi=x}\frac{(1/f)(\xi)}{(\xi-x)^3}$$

$$=\frac{2\left(Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}\right)^2 f(x)-f(x)Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^3}}{f(x)^3}.$$

From the formula

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt},$$

we obtain

$$Res._{\tau=t}\frac{f(x(\tau),y(\tau))}{(\tau-t)^2}=Res._{\xi=x}\frac{f(\xi,y(t))}{(\xi-x)^2}Res._{\tau=t}\frac{x(\tau)}{(\tau-t)^2}$$

$$+Res._{\eta=y}\frac{f(x(t),\eta)}{(\eta-y)^2}Res._{\tau=t}\frac{y(\tau)}{(\tau-t)^2}.$$

For the implicit function theorem for the function $y=f(x)$ of $F(x,y)=0$ satisfying $F(x,f(x))=0$

$$f'(x)=-\frac{F_x(x,y)}{F_y(x,y)},$$

we obtain

$$Res._{\xi=x}\frac{f(\xi)}{(\xi-x)^2}=-Res._{\xi=x}\frac{F(\xi,y)}{(\xi-x)^2}\left(Res._{\eta=y}\frac{F(x,\eta)}{(\eta-y)^2}\right)^{-1}.$$

For the formula

$$f''(x)=-\frac{F_{xx}+2F_{xy}f'+F_{yy}(f')^2}{F_y},$$

we have the correspondent formula.

14 DIVISION BY ZERO CALCULUS ON MULTI-DIMENSIONAL SPACES

In order to make clear the problem, we give firstly a prototype example. We have the identity by the division by zero calculus, for

$$f(z) = \frac{1+z}{1-z}, \quad f(1) = -1.$$

From the real part and imaginary part of the function, we have, for $z = x + iy$

$$\frac{1 - x^2 - y^2}{(1-x)^2 + y^2} = -1, \quad \text{at} \quad (1, 0)$$

and

$$\frac{y}{(1-x)^2 + y^2} = 0, \quad \text{at} \quad (1, 0),$$

respectively. Why the differences do happen?

14.1 Definition of the division by zero calculus for multidimensional spaces

In order to solve this problem, we will give the definition of the division by zero calculus on multidimensional spaces.

Definition of the division by zero calculus for multidimensional spaces. *For an analytic function $g(z)$ on a domain D on $\mathbf{C}^n, n \geq 1$, we set*

$$E = \{z \in D; g(z) = 0\}.$$

For an analytic function $f(z)$ on the set $D \setminus E$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} C_n(z) g(z)^n, \tag{14.1}$$

for analytic functions $C_n(z)$ on D, we define the division by zero calculus by the correspondence

$$f \longrightarrow F_{f,g=0}(z) := C_0(z)$$

that shows a natural analytic function of the function f on the domain D derived from $D \setminus E$ with respect to $E = \{z \in D; g(z) = 0\}$.

Of course, this definition is a natural extension of one dimensional case. The expression (14.1) may be guaranteed by the general Laurent expansion that was introduced by Takeo Ohsawa:

Proposition 13.1 *In the Definition of the division by zero calculus for mutidimensional spaces, if the domain D is a regular domain, for any analytic function g, the expansion* (14.1) *is possible.*

See [119] for the related topics.

However, since the uniqueness of the expansion is, in general, not valid, the division by zero calculus is not determined uniquely. However, we are very interested in the expansion (14.1) and the property of the function $C_0(z)$ as in the one dimensional case.

From the above arguments, we can see the desired results for the examples as follows:

$$\frac{1 - x^2 - y^2}{(1-x)^2 + y^2}$$

$$= -1 + \frac{2(1-x)}{(1-x)^2 + y^2} = -1, \quad \text{at} \quad (1,0)$$

and

$$\frac{y}{(1-x)^2 + y^2} = 0, \quad \text{at} \quad (1,0).$$

14.2 In parameter representations

For example, we will consider the parameter representation

$$x^2 = \frac{a(u-a)(v-a)}{(b-a)(c-a)}, \tag{14.2}$$

$$y^2 = \frac{b(u-b)(v-b)}{(c-b)(a-b)}, \tag{14.3}$$

$$z^2 = \frac{c(u-c)(v-c)}{(a-c)(b-c)}, \tag{14.4}$$

of the ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \quad a, b, c, > 0 \tag{14.5}$$

([21], 112 page).

For the very natural case $b = a$, how will be the parameter representations (13.2)-(13.4)?

At first, we have, by the division by zero, for $b = a$,

$$\frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{c} = 1 \tag{14.6}$$

and

$$x^2 = 0. \tag{14.7}$$

Therefore

$$\frac{y^2}{a} + \frac{z^2}{c} = 1. \tag{14.8}$$

This will mean that (13.5) is the rotation of (13.8) around the z axis and (13.8) is the cut elliptic function of (13.5) by the plane $x = 0$.

Next, by the division by zero calculus, we have

$$y^2 = -\frac{1}{(c-a)^2}[c(u-a)(v-a) - a(c-a)(u-a) - a(c-a)(v-a)]. \tag{14.9}$$

However, since we are considering the case with $x = 0$, the parameters u and v have to be restricted as $u = a$ or $v = a$. We fix as $v = a$ then we have

$$y^2 = \frac{a(a-u)}{a-c}. \tag{14.10}$$

Finally, of course, we have

$$z^2 = \frac{c(u-c)}{a-c}. \tag{14.11}$$

Meanwhile, we will consider the parametric representations with $c = 0$

$$x^2 = \frac{-(u-a)(v-a)}{b-a}, \tag{14.12}$$

$$y^2 = \frac{-(u-b)(v-b)}{a-b}, \tag{14.13}$$

$$z^2 = 0. \tag{14.14}$$

Then, we note that

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 - \frac{uv}{ab} \tag{14.15}$$

and so, for the case $u = 0$ or $v = 0$, we obtain the parameter representation for the elliptic curve

$$\frac{x^2}{a} + \frac{y^2}{b} = 1. \tag{14.16}$$

14.3 Open problems

In Subsection 2, we gave natural interpretations for the parameter representations (13.2)-(13.4) for the case $b = a$ by the division by zero and division by zero calculus. However, we wonder

Open problem: *For the parameter representations (13.2)-(13.4), could we derive some parameter representations of (13.6)?*

When, we use (13.6), (13.4) and (3.9), we have the curious representation:

$$x^2 = \frac{1}{(c-a)^2}[uv(c-a)+ac(u+v)+a^2(u+v+c)-a^3-a(c-a)(v-a)]. \tag{14.17}$$

15 DIVISION BY ZERO CALCULUS IN PHYSICS

We will see the division by zero properties in various physical formulas. We found many and many division by zero phenomena in physics and others, however, we expect many publications about them by the related specialists. As the first stage, here we refer only to elementary formulas, as examples.

15.1 Bhāskara's example – sun and shadow

We will consider the circle such that its center is the origin and its radius R. We consider the point S (sun) on the circle such that $\angle SOI = \theta$; $O(0,0), I(R,0)$. For fixed $d > 0$, we consider the common point $(-L, -d)$ of two line OS and $y = -d$. Then we obtain the identity

$$L = \frac{R\cos\theta}{R\sin\theta} d,$$

([42], page 77.). That is the length of the shadow of the segment of $(0,0) - (0,-d)$ onto the line $y = -d$ of the sun S.

When we consider $\theta \to +0$ we see that, of course

$$L \to \infty.$$

Therefore, Bhāskara considered that

$$\frac{1}{0} = \infty. \tag{15.1}$$

Even nowadays, our mathematics and many people consider so.

However, for $\theta = 0$, we have S=I and we can not consider any shadow on the line $y = -d$, so we should consider that $L = 0$; that is

$$\frac{1}{0} = 0. \tag{15.2}$$

Nothing may be represented by zero; it will be a sense of zero.

Furthermore, for $R = 0$; that is, for S=O, we see its shadow is the point $(0, -d)$ and so $L = 0$ and

$$L = \frac{0 \cos \theta}{0 \sin \theta} d = 0;$$

that is

$$\frac{0}{0} = 0.$$

This example shows that the division by zero calculus is not almighty.

Note that both identities (15.1) and (15.2) are right in their senses. Depending on the interpretations of $1/0$, we obtain INFINITY and ZERO, respectively.

15.1.1 Another example

We consider a triangle ABC with $\overline{AB} = c$, $\overline{BC} = a$, $\overline{CA} = c$. Let x_i be the orthogonal projections of AB and AC to the line BC. Then we have

$$x_i = \frac{1}{2} \left\{ a \mp \frac{(b+c)|b-c|}{a} \right\},$$

([42], pages 70-71.). If $b = c$, then, of course, $x_1 = x_2 = a/2$. For $a = 0$, by the division by zero, we have the reasonable value $x_1 = x_2 = 0$.

15.1.2 Remark

For the example ([42], pages 70-71.), we see that now there is no problem, because we have the relation

$$\frac{R}{j_c} = \frac{r}{R}.$$

Then, we have the right formula

$$y = r \sin \overline{\varphi}.$$

15.2 In balance of a steelyard

We will consider the balance of a steelyard and then we have the equation

$$aF_a = bF_b \tag{15.3}$$

as the moment equality. Here, a, b are the distances from a fixed point and force F_a, F_b points, respectively. Then, we have

$$F_a = \frac{b}{a} F_b.$$

For $a = 0$, should be considered as $F_a = 0$ by the division by zero $b/0 = 0$?

The identity (15.3) appears in many situations, and the above result may be valid similarly.

As a typical case, we recall

Ctesibios (BC. 286-222): *We consider a flow tube with some fluid. Then, when we consider some cut with a plane with its area S and with its velocity v of the fluid on the plane, by continuity, we see that for any cut plane, $Sv = C$; C : constant. That is,*

$$v = \frac{C}{S}.$$

When S tends to zero, the velocity v tends to infinity. However, for $S = 0$, the flow stops and so, $v = 0$. Therefore, this example shows the division by zero $C/0 = 0$ clearly. Of course, in the situation, we have $0/0 = 0$, trivially.

For any nonnegative function $f(x)$ with some parameter x and we will consider that the area S is given by the function

$f(x)$. Then, for some point $f(x_0) = 0$, we will obtain the identity

$$v = \frac{C}{f(x_0)} = 0.$$

We can find many and many similar examples, for example, in Archimedes' principle and Pascal's principle.

We will state one more example:

E. Torricelli (1608 -1646): *We consider some water tank and the initial high $h = h_0$ for $t = 0$ and we assume that from the bottom of the tank with a hole of area A, water is fall down. Then, by the law with a constant k*

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h},$$

we have the equation

$$h(t) = \left(\sqrt{h_0} - \frac{k}{2A}\right)^2.$$

Similarly, of course, for $A = 0$, we have

$$h(t) = h_0.$$

Even the fundamental relation among velocity v, time t and distance s

$$t = \frac{s}{v},$$

we will be able to understand the division by zero

$$\frac{s}{0} = 0$$

and

$$\frac{0}{0} = 0.$$

15.3 By rotation

We will give a simple physical model showing the result $\frac{0}{0} = 0$. We shall consider a disc with $x^2 + y^2 \leq a^2$ rowling uniformly with a positive constant angular velocity ω with its center at the origin. Then we see, at the only origin, $\omega = 0$ and at other all points, ω is a constant. Then, we see that the velocity and the radius r are zero at the origin. This will mean that, in the general formula

$$v = r\omega,$$

or, in

$$\omega = \frac{v}{r}$$

at the origin,

$$\frac{0}{0} = 0.$$

We will not be able to obtain the result from

$$\lim_{r \to +0} \omega = \lim_{r \to +0} \frac{v}{r},$$

because it is the constant.

For a uniform rotation with velocity $\mathbf{v}$ with its center O' and with its radius r. For the angular velocity vector ω and for the moving position P on the circle, we set $\mathbf{r} = OP$. Then,

$$\mathbf{v} = \omega \times \mathbf{r}.$$

If $\omega \times \mathbf{r} = 0$, then, of course, $\mathbf{v} = 0$.

15.4 By the Newton's law

We will recall the fundamental law by Newton:

$$F = G\frac{m_1 m_2}{r^2} \tag{15.4}$$

for two masses m_1, m_2 with a distance r and for a constant G. Of course,

$$\lim_{r \to +0} F = \infty,$$

however, as in our fraction

$$F = 0 = G\frac{m_1 m_2}{0}. \tag{15.5}$$

Of course, here, we can consider the above interpretation for the mathematical formula (15.4) as the new interpretation (15.5). In the ideal case, when two masses are on one point, the force F will not be positive and it will be reduced to zero.

In the Kepler (1571 - 1630) - Newton (1642 - 1727) law for central force movement of the planet,

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GmM}{r^3}\mathbf{r},$$

of course, we have $\mathbf{r} = \mathbf{0}$ for $r = 0$.

For the Coulomb's law, see similar formulas. Indeed, in the formula

$$F = k\frac{(+q)(-q)}{r^2}$$

for $r = 0$, we have $F = 0$.

In general, in the formula

$$F = k\frac{(Q_1)(Q_2)}{r^2}$$

for $r = 0$, we have $F = 0$ (S. Senuma: 2016.8.20.).

Furthermore, as well-known, the bright at a point at the distance r from the origin is given by the formula

$$B = k\frac{P}{r^2},$$

where k is a constant and P is the amount of the light. Of course, we have, at the infinity

$$B = 0.$$

Then, meanwhile, may we consider as

$$B = 0$$

at the origin $r = 0$? Then we can obtain our formula

$$k\frac{P}{0} = 0,$$

as in our new formula.

15.5 An interpretation of $0 \times 0 = 100$ from $100/0 = 0$

The expression $100/0 = 0$ will represent some divisor by the zero in a sense that is not the usual one, and so, we will be able to consider some product sense $0 \times 0 = 100$.

We will show such an interpretation.

We shall consider same two masses m, however, their constant velocities v for the origin are the same on the real line, in the symmetry way. We consider the moving energy product E^2,

$$\frac{1}{2}mv^2 \times \frac{1}{2}m(-v)^2 = E^2.$$

We shall consider at the origin and we assume that the two masses stop at the origin (possible in some case). Then, we can consider, formally

$$0 \times 0 = E^2.$$

The moving energies change to other energies, however, we can obtain some interpretation as in the above.

This example was discovered by M. Yamane presented in the paper [51].

15.6 Capillary pressure in a narrow capillary tube

In a narrow capillary tube saturated with fluid such as water, the capillary pressure is simply expressed as follows,

$$Pc = \frac{2\sigma}{r}$$

where Pc is capillary pressure (suction pressure), σ is surface tension, and r is radius. If r is zero, there is no pressure. However Pc shows infinity, in the common meaning.

This simple equation is based on the Laplace-Young equation

$$P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 and R_2 are two principal radii of curvature at any point on the surface of a droplet or a bubble and in the case of spherical form $R_1 = R_2 = R$. For a spherical bubble the pressure difference across the bubble film is zero since the pressure is the same on both sides of the film. The Laplace-Young equation reduces to

$$\frac{1}{R_1} + \frac{1}{R_2} = 0.$$

On other hand when diameter of a bubble is decreased and becomes $0(R = 0)$, the bubbles collapse and enormous energy is generated. Accumulated free energy in the bubble is released instantaneously.

This example was discovered by M. Kuroda presented in [51].

15.7 Circles and curvature - an interpretation of the division by zero $r/0 = 0$

We consider a solid body called right circular cone whose bottom is a disc with its radius r_2. We cut the body with a disc of radius $r_1 (0 < r_1 < r_2)$ that is parallel to the bottom disc. We denote the distance by d between both discs and R the distance between the top point of the cone and the bottom circle on the surface of the cone. Then, R is calculated by Eko Michiwaki (8 year old daughter of H. Michiwaki) as follows:

$$R = \frac{r_2}{r_2 - r_1} \sqrt{d^2 + (r_2 - r_1)^2},$$

that is called *EM radius*, because by the rotation of the cone on the plane, the bottom circle writes the circle of radius R. We denote by $K = K(R) = 1/R$ the curvature of the circle with its radius R. We fix the distance d. Now note that

$$r_1 \to r_2 \Longrightarrow R \to \infty.$$

This will be natural in the sense that when $r_1 = r_2$, the circle with its radius R becomes a line.

However, the division by zero will mean that when $r_1 = r_2$, the above EM radius formula makes sense and $R = 0$. What does it mean? Here, note that, however, then the curvature $K = K(0) = 0$ by the division by zero calculus; that is, the circle with its radius R becomes a line, similarly. The curvature of a point (circle of radius zero) is zero.

15.8 Vibration

In the typical ordinary differential equation

$$m\frac{d^2x}{dt^2} = -kx,$$

we have a general solution

$$x = C_1 \cos(\omega t + C_2), \quad \omega = \sqrt{\frac{k}{m}}.$$

If $k = 0$, that is, if $\omega = 0$, then the period T that is given by

$$T = \frac{2\pi}{\omega}$$

should be understood as $T = 0$?

In the typical ordinary differential equation

$$m\frac{d^2x}{dt^2} + kx = f \cos \omega t,$$

we have a special solution

$$x = \frac{f}{m} \frac{1}{|\omega^2 - \omega_0^2|} \cos \omega t, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

Then, how will be the case

$$\omega = \omega_0$$

?

For example, for the differential equation

$$y'' + a^2 y = b \cos \lambda x,$$

we have a special solution, with the condition $\lambda \neq a$

$$y = \frac{b}{a^2 - \lambda^2} \cos \lambda x.$$

Then, when $\lambda = a$, by the division by zero calculus, we obtain the special solution

$$y = \frac{bx \sin(ax)}{2a} + \frac{b \cos ax}{4a^2}.$$

15.9 Spring or circut

We will consider a spring with two spring constants $\{k_j\}$ in a line. Then, the spring constant k of the spring is given by the formula

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2},$$

by Hooke's law. We know, in particular, if $k_1 = 0$, then

$$\frac{1}{k} = \frac{1}{0} + \frac{1}{k_2},$$

and by the division by zero,

$$k = k_2,$$

that is very reasonable. In particular, by Hooke's law, we see that

$$\frac{0}{0} = 0.$$

As we saw for the case of harmonic mean, in this case $k_1 = 0$, the zero means that the spring does not exist.

The corresponding result for the case of Ohmu's law is similar and valid.

15.10 Motion

A and B start at the origin on the real positive axis with, for $t = 0$

$$\frac{d^2x}{dt^2} = a, \quad \frac{dx}{dt} = u$$

and

$$\frac{d^2x}{dt^2} = b, \quad \frac{dx}{dt} = v,$$

respectively. After the time T and at the distance X from the origin, if they meet, then we obtain the relations

$$T = \frac{2(u-v)}{b-a}$$

and

$$X = \frac{2(u-v)(ub-va)}{(b-a)^2}.$$

For the case $a = b$, we obtain the reasonable results $T = 0$ and $X = 0$.

We will consider the motion (x, y) represented by $x = \cos\theta, y = \sin\theta$ from $(1, 0)$ to $(-1, 0)$ $(0 \leq \theta \leq \pi)$ with the condition

$$v_x = \frac{dx}{dt} = -\sin\theta\frac{d\theta}{dt} = V \quad \text{(constant)}.$$

Then, we have that

$$v_y = \frac{dy}{dt} = -V\frac{1}{\tan\theta},$$

and

$$a_y = \frac{d^2y}{dt^2} = -V^2 \frac{1}{\sin^3\theta}.$$

Then we see that

$$v_y(1,0) = 0, \text{that is, } \frac{1}{\tan 0} = 0,$$

$$v_y(-1,0) = 0, \text{that is, } \frac{1}{\tan \pi} = 0,$$

$$a_y(1,0) = 0, \text{that is, } \frac{1}{\sin^3 0} = 0,$$

and

$$a_y(-1,0) = 0, \text{that is, } \frac{1}{\sin^3 \pi} = 0.$$

15.11 Darcy's law for fluid through porpous media

Diffusion phenomenon and penetration phenomenon may be represented by the partial differential equations

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u^m}{\partial x^2}$$

for some constants ν and m.

Indeed, density u and pressure p may be related by equation of state

$$u = \gamma p^\alpha,$$

with some constants γ and α.

By the conservative law, we have, for porocity ν and velocity v

$$\frac{\partial(uv)}{\partial x} = -\nu \frac{\partial u}{\partial t}.$$

At the last, by Darcy's law, we have for some constant k

$$v = -k\frac{\partial p}{\partial x}.$$

By chancelling v, p from three equations we obtain

$$\frac{\partial u}{\partial t} = \frac{k}{\nu\gamma(\alpha+1)}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u^{1+1/\alpha}\right)$$

([49], 21-22). As basic references, see ([27, 6, 22]).

Be setting $m = 1 + 1/\alpha$, we have the desired equation.

Note that for $\alpha = 0$, with the division by zero $1/0 = 0$, we have the right differential equation.

Meanwhile, for $\alpha = -1$, by the division by zero calculus, we have

$$\frac{\partial u}{\partial t} = \frac{k}{\nu\gamma}\frac{\partial^2}{\partial x^2}\left(-\log u\right).$$

How will be this partial differential equation?

15.12 RCL and RL circuits

We will consider an RCL circuit stated by the ordinary differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idi = E_0 \sin\omega t, \tag{15.6}$$

$$i = \frac{E_0}{\sqrt{R^2 + ((\omega L - (1/(\omega C))^2}}\sin\left(\omega t - \varphi\right); \tag{15.7}$$

$$\varphi = \arctan\frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right). \tag{15.8}$$

Here, $E_0 \sin\omega t$ is a given AC voltage.

In this circuit, for the case $C = 0$ that is the condenser (capacitor) is missing, we obtain the corresponding result precisely by the division by zero

$$\frac{1}{C} = 0$$

and

$$\frac{1}{\omega C} = 0.$$

Physical interpretation

The condenser C may be realized or represented by the formula

$$C = \frac{\varepsilon S}{d},$$

where S is the surface measure of parallel planes, d is their distance and ε is the physical constant.

If d tends to infinity with fixed S and ε, then C tends to zero. However, then the circuit will be disconnect. Therefore, with this interpretation we can not consider the above typical LR circuit.

Meanwhile, from the expression

$$\frac{1}{C} = \frac{d}{\varepsilon S},$$

if $d = 0$, the circuit becomes just the LR circuit. Then, if our division by zero property $1/0 = 0$ is not admitted, we will not be able to give a suitable interpretation for the LR circuit and the corresponding differential equation.

We can find many and many the division by zero and division by zero calculus in physics.

16 INTERESTING EXAMPLES IN THE DIVISION BY ZERO

We will give interesting examples in the division by zero. Indeed, the division by zero may be looked in the elementary mathematics and also in the universe.

- For the line
$$\frac{x}{a} + \frac{y}{b} = 1,$$
if $a = 0$, then by the division by zero, we have the line $y = b$. This is a very interesting property creating new phenomena at the term x/a for $a = 0$.

  Note that here we can not consider the case $a = b = 0$.

  However, for the equation $bx+ay = ab$, we have the meaning.

- We will consider the line equation, for fixed $a > 0$
$$\frac{x}{-a\cos\alpha} + \frac{y}{a\sin\alpha} = 1.$$
We set: $P(-a\cos\alpha, 0)$ and $Q(0, a\sin\alpha)$, $\overline{PQ} = a$ and $\angle OPQ = \pi - \alpha$, where O is the origin. Then, if $\alpha = 0$, $x = -a$ and if $\alpha = \pi/2$, then $y = a$ and they are reasonable.

- For the area $S(a, b) = ab$ of the rectangle with its sides of lengths a, b, we have
$$a = \frac{S(a, b)}{b}$$
and for $b = 0$, formally
$$a = \frac{0}{0}.$$

However, there exists a contradiction. $S(a, b)$ depends on b and by the division by zero calculus, we have, for the case $b = 0$, the right result

$$\frac{S(a,b)}{b} = a.$$

- For the identity

$$(a^2 + b^2)(a^2 - b^2) = c^2(a^2 - b^2); \quad a, b, c > 0,$$

if $a \neq b$, then we have the Pythagorean theorem

$$a^2 + b^2 = c^2.$$

However, for the case $a = b$, we have also the Pythagorean theorem, by the division by zero calculus

$$2a^2 = c^2.$$

- Let $\alpha_j; j = 1, ..., n$ be the solutions of the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0 = 0, \quad a_n \neq 0,$$

then, $\frac{1}{\alpha_j}; j = 1, ..., n$ are the solution of the equation

$$f\left(\frac{1}{x}\right) = 0,$$

when we apply the division by zero.

- For any $x, y \in \mathbf{R}$ if $f(x + y) = f(x) + f(y)$, then for any positive integer n, we have

$$f\left(\frac{x}{n}\right) = \frac{1}{n} f(x).$$

Then, for $n = 0$, we have

$$f\left(\frac{x}{0}\right) = \frac{1}{0} f(x)$$

Then, for
$$\frac{x}{0} = \frac{1}{0} = 0$$
we have
$$f(0) = 0,$$
that is right.

- For the function
$$ax^2 + bx + c = a(x - \alpha)(x - \beta), \quad a \neq 0,$$
$$\frac{1}{ax^2 + bx + c} = \frac{1}{a(x - \alpha)(x - \beta)}.$$
Then, for $x = \alpha$ for $\alpha \neq \beta$, we have
$$-\frac{1}{a(\alpha - \beta)^2.}$$
For $\beta = \alpha$ and for $x = \alpha$, we have 0.

- In a Hilbert space H, for a fixed member v and for a given number d we set
$$V = \{y \in H; (y, v) = d\}$$
and for fixed $x \in H$
$$d(x, V) := \frac{|(x, v) - d|}{\|v\|}.$$
If $v = 0$, then, $(y, v) = 0$ and d has to be zero. Then, since $H = V$, we have
$$0 = \frac{0}{0}.$$

- For the equation
$$\mathbf{a} \times \mathbf{x} = \mathbf{b},$$

the solutions exist if and only if $\mathbf{a}\cdot\mathbf{b}=0$ and then, we have

$$\mathbf{x}=\frac{\mathbf{b}\times\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}+C\mathbf{a}.$$

For $\mathbf{a}=\mathbf{0}$, we have $\mathbf{x}=\mathbf{0}$ by the division by zero.

For the equation

$$\mathbf{a}\cdot\mathbf{x}=b,$$

we have the equation

$$\mathbf{x}=\frac{b\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}+\mathbf{c}\times\mathbf{a}.$$

If $\mathbf{a}=\mathbf{0}$, then we have $\mathbf{x}=\mathbf{0}$ by the division by zero.

- We consider 4 lines

$$\begin{aligned}
a_1x+b_1y+c_1&=0,\\
a_1x+b_1y+c_1'&=0,\\
a_2x+b_2y+c_2&=0,\\
a_2x+b_2y+c_2'&=0.
\end{aligned}$$

Then, the area S surrounded by these lines is given by the formula

$$S=\frac{|c_1-c_1'|\cdot|c_1-c_1'|}{|a_1b_2-a_2b_1|}.$$

Of course, if $|a_1b_2-a_2b_1|=0$, then $S=0$.

- $\frac{1}{\sin 0}=\frac{1}{\cos\pi/2}=0$. Consider the linear equation with a fixed positive constant a

$$\frac{x}{a\cos\theta}+\frac{y}{a\sin\theta}=1.$$

Then, the results are clear from the graphic meanings.

- For the tangential line at a point $(a\cos\theta, b\sin\theta)$ on the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{16.1}$$

we have $Q(a/(\cos\theta), 0)$ and $R(0, b/(\sin\theta))$ as the common points with x and y axes, respectively. If $\theta = 0$, then $Q(a, 0)$ and $R(0, 0)$. If $\theta = \pi/2$, then $Q(0, 0)$ and $R(0, b)$.

- For the tangential line at the point $(a\cos\theta, b\sin\theta)$ on the elliptic curve (16.1), we shall consider the area $S(\theta)$ of the triangle formed by this line and x, y axes

$$S(\theta) = \frac{ab}{|\sin\theta|}.$$

Then, by the division by zero calculus, we have $S(0) = 0$.

- The common point of B (resp. B') of a tangential line of (16.1) and the line $x = a$ (resp. $x = -a$) is given by

$$B\left(a, \frac{b(1 - \cos\theta)}{\sin\theta}\right).$$

(resp.

$$B'\left(-a, \frac{b(1 + \cos\theta)}{\sin\theta}\right).$$

) The circle with its diameter BB' is given by

$$x^2 + y^2 - \frac{2b}{\sin\theta}y - (a^2 - b^2) = 0.$$

Note that this circle passes two forcus points of the elliptic curve. Note that for $\theta = 0$, we have the reasonable result, by the division by zero calculus

$$x^2 + y^2 - (a^2 - b^2) = 0.$$

In the classical theory for quadratic curves, we have to arrange globally it by the division by zero calculus.

- On the real line, we look at the point P with angle α and β with a distance l. Then, the high of the point P is given by
$$h = \frac{l \sin\alpha \sin\beta}{\sin(\alpha - \beta)}.$$
Then, if $\alpha = \beta$, then, by the division by zero, $h = 0$.

- We consider two tangential lines from a point A for a circle C and another line with two common points P and Q with C in the way A-P-Q. Let B be the common point with the line and two tangential points in the way A-P-B-Q. Then, we know the identity
$$\frac{AP}{PB} = \frac{AQ}{QB}$$
or
$$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}.$$
These identities are valid even if P=B=Q with
$$\frac{AP}{0} = \frac{AQ}{0} = 0.$$
Similarly, we consider two chords AB and CD of a circle with a common point P. Then we have
$$\frac{PA}{PC} = \frac{PD}{PB}.$$
If $C = P = B$, then we have
$$\frac{AP}{0} = \frac{PD}{0} = 0.$$

- On the complex plane, the points $\{z_j; j = 1, 2, 3, 4\}$ on a circle if and only if
$$\frac{2}{z_1 - z_2} = \frac{1}{z_1 - z_2} + \frac{1}{z_1 - z_4}.$$

If $z_1 = z_2$, then we have, by the division by zero

$$z_1 = \frac{z_1 + z_2}{2}.$$

For 4 points $\{z_j; j = 1, 2, 3, 4\}$ on the complex plane, let θ be the angle for the lines $z_1 z_2$ and $z_3 z_4$. Then, we have

$$\cos\theta = \frac{1}{2}\frac{(z_2 - z_1)\overline{(z_4 - z_3)} + \overline{(z_2 - z_1)}(z_4 - z_3)}{|z_2 - z_1||z_4 - z_3|}.$$

If $z_1 = z_2$, then we have, by the division by zero, $\theta = \pi/2$.

- We see that z belongs to the closed convex hull of $\{z_1, z_2, ..., z_n\}$ if and only if

$$\frac{\lambda_1}{z - z_1} + \frac{\lambda_2}{z - z_2} + ... + \frac{\lambda_n}{z - z_n} = 0$$

for

$$\lambda_j \geq 0; \quad \sum_{j=1}^{n} \lambda_j^2 \neq 0.$$

For $n = 1$, the statement is valid with the division by zero

$$\frac{\lambda_1}{z - z_1} = 0$$

and

$$z = z_1, \frac{\lambda_1}{0} = 0.$$

Furthermore, for $z = z_1$ we have the very interesting identity

$$\frac{\lambda_2}{z_1 - z_2} + ... + \frac{\lambda_n}{z_1 - z_n} = 0.$$

- We understand, by the division by zero, for the function

$$f(z) = \frac{z}{|z|},$$

$$f(0) = \frac{0}{|0|} = \frac{0}{0} = 0,$$

as in the sign function.

- We fix the lines $y = d$ and $x = L$ $(d, L > 0)$. We consider a line through two points $(0, t); t > d$ and (L, d), and let D be the common point with the line and the x axis. Then, we have the identity

  $$\frac{D}{L} = \frac{t}{t-d}.$$

  When $t = d$, by the division by zero, from $d/0 = 0$ we have $D = 0$ which is reasonable in our new mathematics. However, from the identity

  $$\frac{t}{t-d} = 1 + \frac{d}{t-d},$$

  by the division by zero calculus, we have another reasonable result $D = L$.

- We consider the line through the fixed point P$(-1, 1)$ and Q$(x, 0), x \geq 0$ and the mapping

  $$x = \frac{-a}{a-1} = -1 - \frac{1}{a-1}.$$

  This function maps $[0, 1)$ in the y axis onto $[0, \infty)$ on the x axis in one to one way. For $a = 1$, by the division by zero calculus, we have $x = -1$ and so the function maps $[0, 1]$ in the y axis onto $\{-1\} \cup [0, \infty)$ on the x axis in one to one way.

  Then, note that for

  $$L(x) = \sqrt{1 + (1 + x)^2}$$

  we have

  $$L(-1) = 1.$$

- We recall the Bramaguputa (598-668?) theorem. We assume that for points A,B,C,D on a circle, $AB = a, BD = d, CD = c, DA = b$. Let P be the common point of the lines AB and DC, and we set $BP = e, CP = f$. Then, we have
$$e = \frac{dcb + ad^2}{(b-c)(b+d)}$$
and
$$f = \frac{abd + cd^2}{(b-c)(b+d)}.$$
If $b = c$, then we have $e = f = 0$.

  Meanwhile, on the complex z plane, for four points $A(a)$, $B(b), C(c), D(d)$ on a circle $|z| = r$. Let $P(p)$ be the common point of the lines AB and CD. Then, we have
$$p = \frac{bcd + acd - abd - abc}{cd - ab}.$$
Note that for $cd - ab = 0$, we have
$$p = \frac{0}{0} = 0.$$

- The area $S(x)$ surrounded by two x, y axes and the line passing a fixed point $(a, b), a, b > 0$ and a point $(x, 0)$ is given by
$$S(x) = \frac{bx^2}{2(x-a)}.$$
For $x = a$, we obtain, by the division by zero calculus, the very interesting value
$$S(a) = ab.$$

- For example, for fixed point $(a, b); a, b > 0$ and fixed a line $y = (\tan\theta)x, 0 < \theta < \pi$, we will consider the line $L(x)$ passing two points (a, b) and $(x, 0)$. Then, the area $S(x)$

of the triangle surrounded by three lines $y = (\tan\theta)x$, $L(x)$ and the x axis is given by

$$S(x) = \frac{b}{2}\frac{x^2}{x-(a-b\cot\theta)}.$$

For the case $x = a - b\cot\theta$, by the division by zero calculus, we have

$$S(a-b\cot\theta) = b(a-b\cot\theta).$$

Note that this is the area of the parallelogram through the origin and the point (a,b) formed by the lines $y = (\tan\theta)x$ and the x axis.

- We consider the circle

$$h(x^2+y^2)+(1-h^2)y-h=0$$

through the points $(-1,0)$, $(1,0)$ and $(0,h)$. If $h=0$, then we have

$$y = 0.$$

However, from the equation

$$x^2+y^2+\left(\frac{1}{h}-h\right)y-1=0,$$

by the division by zero, we have an interesting result

$$x^2+y^2=1.$$

- We consider the regular triangle with the vertices

$$(-a/2,\sqrt{3}a/2),(a/2,\sqrt{3}a/2).$$

Then, the area $S(h)$ of the triangle surrounded by the three lines that the line through $(0,h+\sqrt{3}a/2)$ and

$$(-a/2,\sqrt{3}a/2),$$

the line through $(0, h+\sqrt{3}a/2)$ and $(a/2, \sqrt{3}a/2)$ and the x- axis is given by

$$S(h) = \frac{\left(h + (\sqrt{3}/2)a\right)^2}{2h}.$$

Then, by the division by zero calculus, we have, for $h = 0$,

$$S(0) = \frac{\sqrt{3}}{2}a^2.$$

- Similarly, we will consider the cone formed by the rotation of the line

$$\frac{kx}{a(k+h)} + \frac{y}{k+h} = 1$$

and the x, y plane around the z- axis ($a, h > 0$, and a, h are fixed). Then, the volume $V(x)$ is given by

$$V(k) = \frac{\pi}{3}\frac{a^2(k+h)^3}{k^2}.$$

Then, by the division by calculus, we have the reasonable value

$$V(0) = \pi a^2 h.$$

- For the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n,$$

we have, of course, $\lim_{n\to\infty} a_n = e$. Meanwhile, by formally, we have

$$a_0 = \left(1 + \frac{1}{0}\right)^0 = 1^0 = 1.$$

However, we obtain

$$a_0 = \exp\left\{n \log\left(1 + \frac{1}{n}\right)\right\}_{n=0} = e, \tag{16.2}$$

by the division by zero calculus. Indeed, for $x = 1/n$, we have

$$n \log\left(1 + \frac{1}{n}\right) = \frac{1}{x}\left(x - \frac{x^2}{2} + ...\right)$$

and this equals 1 for the point at infinity, by the division by zero calculus. Note that for the definition by exponential functions by (16.2) is fundamental.

- For example, for the plane equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

for $a = 0$, we can consider the line naturally, by the division by zero

$$\frac{y}{b} + \frac{z}{c} = 1.$$

- For the Gauss map function

$$f(x) = \frac{1}{x} - \left[\frac{1}{x}\right],$$

we have, automatically, by the division by zero

$$f(0) = 0.$$

- For the product and sum representations

$$\Pi_{\nu=-\infty,\nu\neq 0}^{\infty}\left(1 - \frac{z}{\nu\pi}\right)\exp\frac{z}{\nu\pi}$$

and

$$\sum_{\nu=-\infty,\nu\neq 0}^{\infty}\left(\log\left(1 - \frac{z}{\nu\pi}\right) + \frac{z}{\nu\pi}\right),$$

we do not need the conditions $\nu \neq 0$, because, the corresponding terms are automatically 1 and zero, respectively, by the division by zero.

- Let X and Y be norm spaces and T be a bounded linear operator from X to Y. Then, its norm is given by

$$||T|| = \sup_{x \neq 0} \frac{||Tx||}{||x||}.$$

However, if $x = 0$, then $Tx = 0$ and so, for $x = 0$,

$$\frac{||Tx||}{||x||} = 0.$$

Therefore, we do not need the condition $x \neq 0$ in the definition.

- For $a_j > 0$, we have the inequality

$$(a_1 + a_2 + \cdots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2.$$

If $a_n = 0$, by the division by zero, the inequality holds for $n - 1$.

- For the harmonic numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$$= \frac{1}{0} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

we have

$$H_n = H_{n-1} + \frac{1}{n}.$$

Then, $H_1 = 1$ and $H_2 = 3/2$, and we obtain

$$H_0 = \frac{1}{0} = 0$$

(M. Cervnka: 2017.9.22.).

- We consider the Weierstrass function

$$E(z,q) = (1-z)\exp\left(0 + z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + \frac{1}{q}z^q\right).$$

For $q = 0$, from

$$\frac{1}{q}z^q = 0,$$

we obtain automatically

$$E(z,0) = 1 - z.$$

- In the Fermat theorem that for a prime number p, a is an integer with no common integer with p, then

$$a^{p-1} \equiv 1$$

with mod p, from

$$\frac{1}{a} \equiv a^{p-2}$$

with mod p, we have formally

$$\frac{1}{0} \equiv 0$$

with mod p (M. Cervnka: 2017.9.22.).

- For the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

of the quadratic equation

$$ax^2 + bx + c = 0,$$

we have, the solution, for $a = 0$ and $b \neq 0$,

$$x = -\frac{c}{b},$$

by the division by zero calculus. For this viewpoint we see the following representation of the general solution

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}.$$

Meanwhile, V. V. Puha got the representation

$$x = \frac{c}{b}\left(\frac{a}{a} - 1\right) + \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(V. V. Puha: 2018.6.9.5:28).

- Let X be a nonnegative random variable with a continuous distribution F, then the mean residual life function $M(x)$ is given by, if $1 - F(x) > 0$,

$$M(x) = \frac{\int_x^\infty (1 - F(\xi))d\xi}{1 - F(x)}.$$

However, if $1 - F(x) = 0$, automatically, we have $M(x) = 0$, by the division by zero.

- As in the line case, in the hyperbolic curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0, \tag{16.3}$$

by the representations by parameters

$$x = \frac{a}{\cos\theta} = \frac{a}{2}\left(\frac{1}{t} + t\right)$$

and

$$y = \frac{b}{\tan\theta} = \frac{b}{2}\left(\frac{1}{t} - t\right),$$

the origin $(0, 0)$ may be included as the point of the hyperbolic curve, as we see from the cases $\theta = \pi/2, 0$ and $t = 0$.

In addition, from the fact, we will be able to understand that the asymptotic lines are the tangential lines of the hyperbolic curve.

The two tangential lines of the hyperbolic curve with gradient m is given by

$$y = mx \pm \sqrt{a^2m^2 - b^2} \tag{16.4}$$

and the gradients of the asymptotic lines are

$$m = \pm\frac{b}{a}.$$

Then, we have asymptotic lines $y = \pm\frac{b}{a}x$ as tangential lines.

The common points of (16.3) and (16.4) are given by

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2m}{\sqrt{a^2m^2 - b^2}}\right).$$

For the case $a^2m^2 - b^2 = 0$, they are $(0, 0)$.

- We will consider a general cone quadratic curve. For this purpose, we will consider the line $x = -k, (k > 0)$ and a fixed point $O = F(0, 0)$. Then a general cone curve may be represented as

  $$\frac{\overline{PF}}{\overline{PH}} = \frac{r}{k + \cos\theta} = e$$

  with $H(-k, y)$, $P = (x, y)$, $\overline{PF} = r$ and $\theta = \angle PFE$, $E = (1, 0)$.

  Then, we will consider the case $e = 0$. Of course, then the origin may be a point circle at the origin. Meanwhile, for $k + r\cos\theta = 0$; that is, we have the line $x = -k$. These two cases may be considered as a family of cone quadratic curves.

- We consider the surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad a, b, c > 0.$$

For the parameter representations

$$\frac{y}{b} + \frac{z}{c} = \lambda\left(1 + \frac{x}{a}\right), \frac{y}{b} - \frac{z}{c} = \frac{1}{\lambda}\left(1 - \frac{x}{a}\right)$$

and

$$\frac{y}{b} + \frac{z}{c} = \mu\left(1 - \frac{x}{a}\right), \frac{y}{b} - \frac{z}{c} = \frac{1}{\mu}\left(1 + \frac{x}{a}\right),$$

we do not need to assume that $\lambda, \mu \neq 0$.

- We fix a circle

$$x^2 + (y - a)^2 = a^2, \quad a > 0.$$

At the point $(2a + d, 0), d > 0$, we consider two tangential lines for the circle. Let 2θ is the angle between two tangential lines at the point $(2a + d, 0)$, Then, the area $S(h) = S(\theta)$ and the length $L(x) = L(\theta)$ are given by

$$S(h) = S(\theta) = \frac{a}{\sqrt{h}}(h + 2a)^{\frac{3}{2}}$$

$$= \frac{a^2}{\cos\theta}\left(\sin\theta + 2 + \frac{1}{\sin\theta}\right)$$

and

$$L(h) = L(\theta) = \frac{a}{\sqrt{h}}\sqrt{h + 2a}$$

$$= a\left(\frac{1}{\cos\theta} + \tan\theta\right),$$

respectively. For $h = 0$ and $\theta = 0$, by the division by zero calculus, we see that all are zero.

- We consider two spheres defined by

$$x^2 + y^2 + z^2 + 2a_j x + 2b_j y + 2c_j z + 2d_j = 0, \quad j = 1, 2.$$

Then, the angle θ by two spheres is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2 - (d_1 + d_2)}{\sqrt{a_1^2 + b_1^2 + c_1^2 - 2d_1}\sqrt{a_2^2 + b_2^2 + c_2^2 - 2d_2}}.$$

If two spheres are orthogonal or one sphere is a point sphere, then $\cos\theta = 0$.

- For the parabolic equation

$$y^2 = 4px,$$

two points $(pt^2, 2pt)$ and $(ps^2, 2ps)$ is a diameter if and only if

$$(s - t)\{t(s + t) + 2\} = 0; \quad s = -t - \frac{2}{t}$$

and the diameter r is given by

$$r^2 = p^2(t - s)^2\{(t + s)^2 + 4\}.$$

Here, we should consider the case $t = s = 0$ as $r = 0$ and

$$0 = -0 - \frac{2}{0},$$

and the x and y axes are the orthogonal two tangential lines of the parabolic equation.

- For the parameter equation

$$x = t - \frac{1}{t},$$

$$y = t^2 + \frac{1}{t^2}$$

we have

$$y = x^2 + 2.$$

For $t = 0$, we have $x = y = 0$.

For the function

$$y^2 = 4x + 4,$$

for $x = r\cos\theta, y = r\sin\theta$ we have

$$r = \frac{2}{1 - \cos\theta}.$$

Note that for $\theta = 0$, $r = 0$.

- For the integral equation, for a constant k

  $$\int_0^x y dx = ky,$$

  we have the general solution

  $$y = C \exp\frac{x}{k}.$$

  If $k = 0$, then, of course, we have, $y = C$.

  For the integral equation

  $$\int_0^x y dx = k \int_0^x \sqrt{1 + (y')^2} dx$$

  we have the solution

  $$y = \frac{k}{2}\left\{\exp\frac{x}{k} + \exp\left(-\frac{x}{k}\right)\right\}.$$

  If $k = 0$, then, we should have $y = 0$.

- We consider the Cayley transform; for

  $$A = \begin{pmatrix} 0 & \cot\frac{\theta}{2} \\ -\cot\frac{\theta}{2} & 0 \end{pmatrix}$$

and

$$U = (A+E)(A-E)^{-1} =$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then, for $\theta = \pi$, we have the right result, by the division by zero calculus as in

$$A = O$$

and

$$U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

17 WHAT IS THE ZERO?

The zero 0 as the complex number or real number is given clearly by the axioms by the complex number field and real number field, respectively. For this fundamental idea, we should consider the **Yamada field** containing the division by zero. The Yamada field and the division by zero calculus will arrange our mathematics, beautifully and completely; this will be our real and complete mathematics.

Standard value

The zero is a center and stand point (or bases, a standard value) of the coordinates - here we will consider our situation on the complex or real 2 dimensional spaces. By stereographic projection mapping or the Yamada field, the point at infinity 1/0 is represented by zero. The origin of the coordinates and the point at infinity correspond with each other.

As the standard value, for the point $\omega_n = \exp\left(\frac{\pi}{n}i\right)$ on the unit circle $|z| = 1$, for $n = 0$:

$$\omega_0 = \exp\left(\frac{\pi}{0}i\right) = 1, \quad \frac{\pi}{0} = 0.$$

For the mean value

$$M_n = \frac{x_1 + x_2 + ... + x_n}{n},$$

we have

$$M_0 = 0 = \frac{0}{0}.$$

Fruitful world

For example, in very general partial differential equations, if the coefficients or terms are zero, we have some simple differential equations and the extreme case is all terms zero; that is, we have the trivial equation $0 = 0$; then its solution is zero. When we consider the converse, we see that the zero world is a fruitful one and it means some vanishing world. Recall the Yamane phenomena, the vanishing result is very simple zero, however, it is the result from some fruitful world. Sometimes, zero means void or nothing world, however, it will show some change as in the Yamane phenomena.

From 0 to 0; 0 means all and all are 0

As we see from our life figure, a story starts from the zero and ends to the zero. This will mean that 0 means all and all are 0, in a sense. The zero is a mother of all.

Note that all the equations are stated as equal zero; that will mean that all are represented by zero in a sense.

Impossibility

As the solution of the simplest equation

$$ax = b \tag{17.1}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (17.1) is impossible. We saw for different parallel lines or different parallel planes, their common point is the origin. Certainly they have the common point of the point at infinity and the point at infinity is represented by zero. However, we can understand also that they have no solutions, no common points, because the point at infinity is an ideal point.

We will consider the point P at the origin with starting at the time $t = 0$ with velocity $V > 0$ and the point Q at the point $d > 0$ with velocity $v > 0$. Then, the time of coincidence P=Q is given by

$$T = \frac{d}{V - v}.$$

When $V = v$, we have, by the division by zero, $T = 0$. This zero represents impossibility. We have many such situations.

We will consider the simple differential equation

$$m\frac{d^2x}{dt^2} = 0, m\frac{d^2y}{dt^2} = -mg \tag{17.2}$$

with the initial conditions, at $t = 0$

$$\frac{dx}{dt} = v_0 \cos\alpha, \quad \frac{dy}{dt} = v_0 \sin\alpha; \quad x = y = 0.$$

Then, the highest high h, arriving time t, the distance d from the starting point at the origin to the point $y(2t) = 0$ are given by

$$h = \frac{v_0^2 \sin\alpha}{2g}, \quad d = \frac{v_0^2 \sin 2\alpha}{g}$$

and

$$t = \frac{v_0 \sin\alpha}{g}.$$

For the case $g = 0$, we have $h = d = t = 0$. We considered the case that they are infinity; however, our mathematics means zero, which shows impossibility.

These phenomena were looked in many cases on the universe; it seems that God does not like the infinity.

As we stated already in the Bhāskara's example – sun and shadow

Zero represents void or nothing

On ZERO, the authors S. K. Sen and R. P. Agarwal [113] published its history and many important properties. See also R. Kaplan [50] and E. Sondheimer and A. Rogerson [115] on the very interesting books on zero and infinity.

India has a great tradition on ZERO, VOID and INFINITY and they are familiar with those concepts.

Meanwhile, Europian (containing the USA) people do not like such basic ideas and they are not familiar with them.

18 CONCLUSION

Apparently, the common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero.

This book is elementary on our division by zero as the first publication of books for the topics. The contents have wide connections to various fields beyond mathematics. The author expects the readers to write some philosophy, papers and essays on the division by zero from this simple source book.

The division by zero theory may be developed and expanded greatly as in the author's conjecture whose break theory was recently given surprisingly and deeply by Professor Qi'an Guan [40] since 40 years proposed in [101] (the original is in [100]).

We have to arrange globally our modern mathematics with our division by zero in our undergraduate level.

We have to change our basic ideas for our space and world.

We have to change globally our textbooks and scientific books on the division by zero.

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