Conformal Mapping from Sphere to Horn Torus, from Plane to Horn Torus and vice versa both

Wolfgang W. Däumler, Perouse, Germany
contact: artmetic@gmx.de

January 1, 2019

Abstract: For certain applications, especially regarding fundamental physical questions, it might be useful to replace the well-known and well-established Riemann sphere by the geometric figure horn torus, which comprises much more properties, complexity and capabilities than the sphere provides. In this paper I will describe the analytical method for respective conformal mappings.

Key Words: conformal mapping, Riemann sphere, horn torus, complex analysis, infinity

Acknowledgement: I want to thank Vyacheslav Puha, Murmansk, Russia, for the idea and the stimulus to search for this conformal mappings and Professor Saburou Saitoh, Kiryu, Japan for the kind inclusion of the topic into several of his papers and presentations.

1 Riemannian Stereographic Projection

There surely is no need to discuss the method and properties of mapping points, lines and curves (functions) from the complex plane onto the Riemannian unit sphere. The stereographic projection with all its properties and associated mathematical laws is sufficiently known. Here we only designate the following elements for figure 1 on next page:

- M: centre of the Riemann sphere
- 0: common point of plane and sphere (touch point, zero point of complex plane)
- N: 'north pole' of sphere on extended line $\overline{0M}$ (opposite 0)
- Z: point on complex plane, assigned to the complex number z
- $|z|$: absolute value of z and length of line $\overline{0Z}$
- P: intersection point of line $\overline{NZ}$ on surface of sphere
- α: central angle 0MP for chord $\overline{0P}$

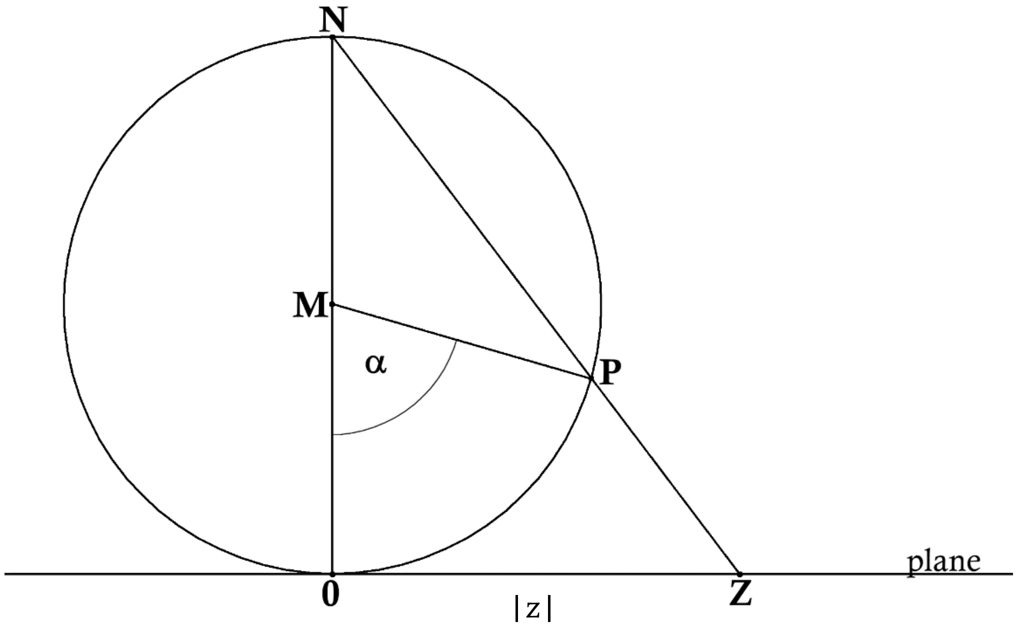

fig. 1

Because of angle 0NZ = $\alpha/2$ (circumferential angle for chord $\overline{0P}$) and $\overline{0N} = 1$ (definition of Riemann sphere) we have for the mappings from plane to sphere and vice versa:

$$\alpha = 2 \tan^{-1}(|z|) \tag{1.1}$$

$$|z| = \tan(\alpha/2) \tag{1.2}$$

2 Mapping from Sphere to Horn Torus and vice versa

Probably no geometrical method like the Riemannian stereographic projection does exist for these mappings, when conformality likewise is stipulated, and so we study sphere and horn torus separately, not for the representation of complex numbers only but for all cases and generally valid.

Therefor we don't declare any appropriate stereographic projection and try to proof the conformality afterwards but we use the conditions of conformality instead to compile and establish the wanted mapping analytically:

One condition for conformal mapping is that small circles on the origin are mapped as small circles on the target surface and therefore we construct small circles on both surfaces that only are dependent on the position of points P on the surface, i.e. only dependent on the angles $\alpha = \angle NMP$ for the sphere (see fig. 2) resp. $\varphi = \angle SMP$ for the horn torus (fig. 3). Due to rotation symmetry of both figures the rotation angle ω doesn't play any role and is arbitrary, and likewise, because of mirror symmetry, α can be exchanged by $\pi - \alpha$ (what corresponds to the exchange of north and south pole of the sphere) and φ can be exchanged by $2\pi - \varphi$. Even the radii r of both figures turn out to be arbitrary and independent.

Following drawings show longitudes, spacing 10°, and latitudes, spacing 20°, P located at angles $\alpha = 50°$ resp. $\varphi = 140°$, $\omega = 35°$ on both figures:

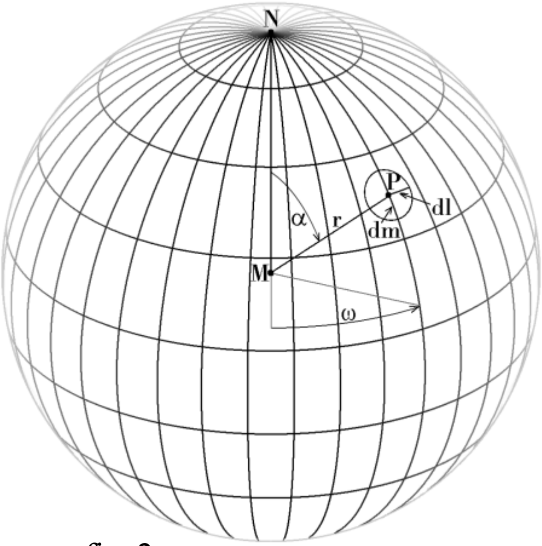

fig. 2

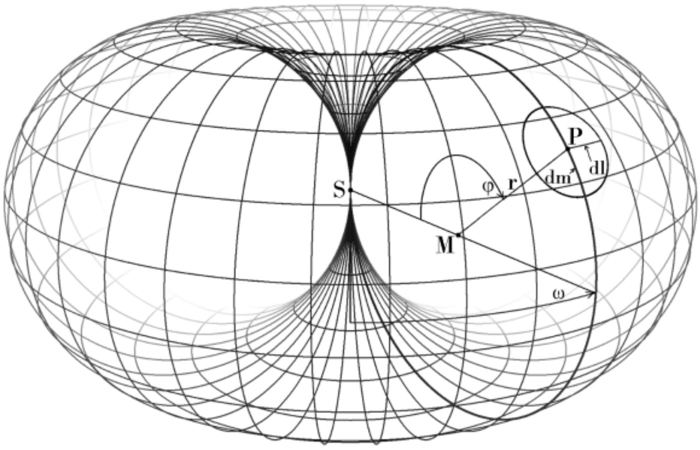

fig. 3

We consider the small circle around any point P and state the condition that the radii dm (in direction of meridians) and dl (parallel to latitudes) have to be equal.
Lengths m of longitudes (meridians) on both figures, sphere and horn torus, are

$$m = 2\pi \cdot r$$

Lengths l of latitudes are computed differently (see supplement for derivation):

$$l = 2\pi \cdot r \cdot \sin\alpha \tag{2.1}$$

on the sphere and

$$l = 2\pi \cdot r \cdot (1 - \cos\varphi) \tag{2.2}$$

on the horn torus, (5.1) applied.

The differentials dm - radii of the respective small circles on the longitude - are

$$dm = r \cdot d\alpha$$

on the sphere and

$$dm = r \cdot d\varphi$$

on the horn torus.

The differentials dl - radii of the respective small circles on the latitude - are

$$dl = d\omega \cdot r \cdot \sin\alpha$$

on the sphere and

$$dl = d\omega \cdot r \cdot (1 - \cos\varphi)$$

on the horn torus.

After equalling dm and dl in both figures separately and cancelling r one has

$$d\alpha = d\omega \cdot \sin\alpha$$

for the sphere and

$$d\varphi = d\omega \cdot (1 - \cos\varphi)$$

for the horn torus.

By solving both equations to dω and equalling we get the differential equation

$$d\alpha/\sin\alpha = d\varphi/(1 - \cos\varphi) \tag{2.3}$$

and finally, by integration, we obtain the condition for conformal mapping

$$\int(1/\sin\alpha)d\alpha \;=\; \int(1/(1 - \cos\varphi))d\varphi \tag{2.4}$$

$$\ln(|\tan(\alpha/2)|) \;=\; - \cot(\varphi/2) + C \tag{2.5}$$

mapping sphere → horn torus:

$$\varphi \;=\; 2 \cot^{-1}(-\ln(|\tan(\alpha/2)|) - C) \tag{2.6}$$

mapping horn torus → sphere:

$$\alpha \;=\; 2 \tan^{-1}(e^{-\cot(\varphi/2) + C}) \tag{2.7}$$

$$0 < \alpha < \pi,\ 0 < \varphi < 2\pi,\ C \text{ any real number}$$

C is a kind of 'zoom/diminishing factor' for the mapped figures and shifts them: case $\alpha \to \varphi$: φ moves towards 2π with increasing $C > 0$, towards 0 with $C < 0$, case $\varphi \to \alpha$: α moves towards π with increasing $C > 0$, towards 0 with $C < 0$, conformality is given for $C \neq 0$ as well, i.e. there is an infinite set of solutions, but mappings are not bijective, when the constant $C \neq 0$ is the same in the inverse mapping.

3 Generalised Riemannian Conformal Mapping

The Riemannian stereographic projection likewise is a special case amongst others, the generalised formulas for conformal mapping plane ↔ sphere, replacing (1.1) and (1.2), but then with loss of bijectivity when $C \neq 1$, are

$$\alpha = 2 \tan^{-1}(C \cdot |z|) \tag{3.1}$$

$$|z| = \tan(\alpha/2) / C \tag{3.2}$$

Real number $C > 0$ again is a kind of 'zoom/diminishing factor' for the mapped figures.

4 Mapping from Plane to Horn Torus and vice versa

When we combine (1.2) with (2.6) resp. (2.7) we get the formulas for conformal mapping direct from plane to horn torus and vice versa:

$$\varphi = 2 \cot^{-1}(-\ln(|z|) - C) \tag{4.1}$$

$$|z| = e^{-\cot(\varphi/2) + C} \tag{4.2}$$

5 Supplement: Length of Horn Torus Latitude

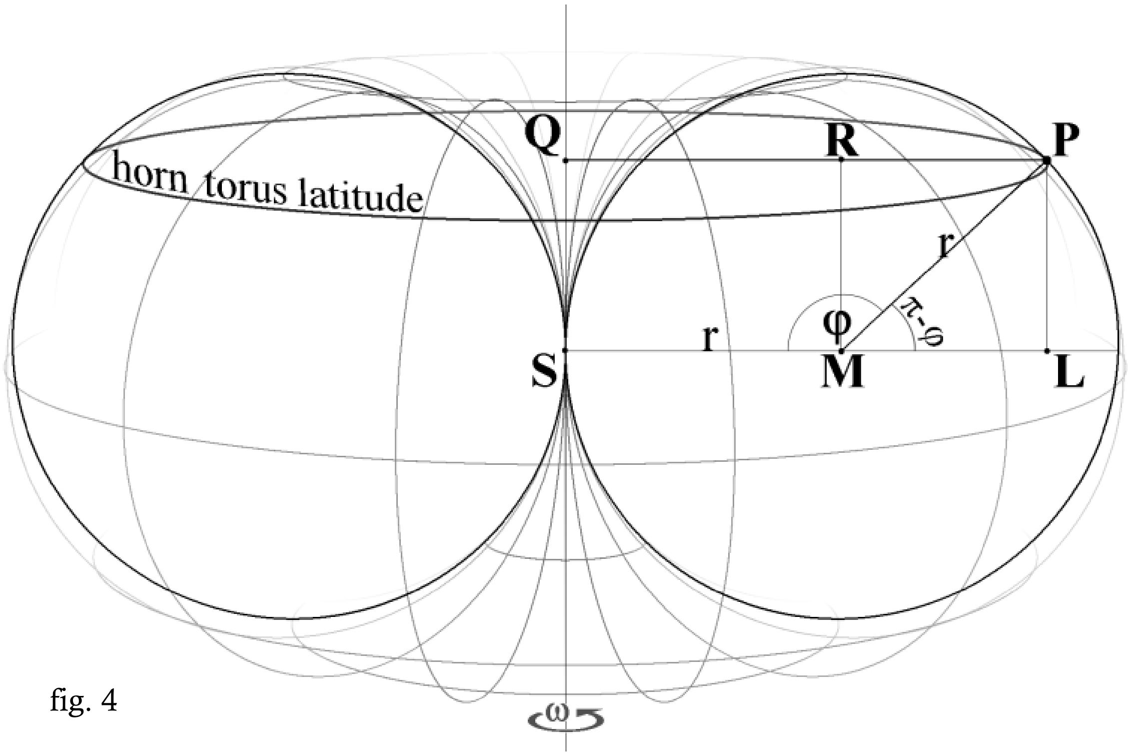

fig. 4

The sketch shows details of a horn torus cross section, embedded in a slightly tilted perspective view, point P is positioned on longitude ω (rotation angle) 90° and latitude φ (torus bulge revolution angle) 135°, S is centre of horn torus, M is centre of circle (half of cross section) with radius r, Q is centre of selected latitude through P and lies on main symmetry axis, auxiliary line $\overline{MR}$ is perpendicular to $\overline{QP}$, auxiliary line $\overline{PL}$ perpendicular to diameter of horn torus cross section circle through S and M. With these points and parameters, we easily can calculate length l of the latitude with radius $\overline{QP}$:

$$\overline{QP} = \overline{QR} + \overline{RP} = \overline{SM} + \overline{ML} = r + r\cdot\cos(\pi - \varphi) = r - r\cdot\cos\varphi = r\cdot(1 - \cos\varphi)$$

$$l = 2\pi\cdot\overline{QP}$$

$$l = 2\pi\cdot r\cdot(1 - \cos\varphi) \tag{5.1}$$

6 Properties of the Horn Torus

For centuries science ignored the existence of the geometric figure horn torus completely or neglected its relevance unduly. Nearly nobody realized, described and applied the exceptional topology, maximum symmetry, high complexity and creative capabilities of this unique object. Now, in recent years, it increasingly appears in publications, mostly in context of particle and quantum physics and in connection to cosmological questions. The subject is developing and promises to stay exciting for a good while.

Horn torus properties get most thrilling when we add dynamic to the figure, when we let it turn, around the main symmetry axis and around the torus bulge, when we let it change its size during the turns, when we combine all these motions and when we finally interlace two, more or even infinitely many such dynamic horn tori with one another.

In respect of the mapping from the sphere one realizes that on the horn torus the points for zero and for infinity merge to one single point, what induces a strange topology. This property is used by Professor Saburou Saitoh in connection with his results in the division by zero issue: http://vixra.org/abs/1904.0052 and several publications more on viXra.org

* * *

The author of this present paper provides a lot of thoughts, texts, images and animations concerning the 'horn torus model' as kind of intellectual game, first to exercise imaginative power, to improve abstract thinking and to generate lots of aha moments, then as proposal for a different approach to physical questions, and finally as part of an interdisciplinary art project on his private website:

https://www.horntorus.com

cite this paper as:
Daeumler, Wolfgang, private publication 2019, https://www.horntorus.com/manifolds/conformal-mapping.pdf

extended version (2021):
https://www.horntorus.com/manifolds/paper/horn-torus-synopsis.pdf
(published at https://romanpub.com/dbzc-vol-1--2021.php)